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On integro quartic spline interpolation

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ABSTRACT

In this paper, we use quartic B-spline to construct an approximating function to agree with the given integral values of a univariate real-valued function over the same intervals. It is called integro quartic spline interpolation. Our interpolation method is new and easy to implement. Moreover, it can work successfully even without any boundary conditions. The interpolation errors are studied. The super convergence (sixth order and fourth order, respectively) in approximating function values and second-order derivative values at the knots is proved. Numerical examples illustrate that our method is very effective and our integro-interpolating quartic spline has higher approximation ability than others.

1. Introduction

In the traditional interpolation problems of numerical analysis, we are often given the function values $y_j = y(x_j)$ at a set of distinct knots x_j over an interval $[a, b]$. We use the given data to determine the so-called interpolating function $p(x)$ such that

$$p(x_j) = y_j = y(x_j), \quad (j = 0, 1, \dots, n).$$

Generally, spline functions, which are well known as piecewise polynomials pieced together at the knots by certain smoothness conditions, are often applied on this topic [1–5].

In this paper, we study a different interpolation problem. We assume that the function values at the knots are not given, but the integral values I_j of the function $y(x)$ on the subintervals $[x_j, x_{j+1}]$ $(j = 0, 1, \dots, n-1)$ are known. Our task is to determine an integro-interpolating function $p(x)$ such that

$$\int_{x_j}^{x_{j+1}} p(x)dx = I_j = \int_{x_j}^{x_{j+1}} y(x)dx, \quad (j = 0, 1, \dots, n-1).$$

Obviously, it is a generalization for the traditional interpolation problem. Furthermore, it has many practical applications in the fields of mechanics, mathematical statistics, numerical analysis, electricity, climatology, oceanography and so on; see [6–11].

Similarly, spline functions can also be used to deal with the new interpolation problems. However, there have been only a few research papers. For example, the integro cubic spline methods over a uniform partition were studied in [11,12], but their error orders are lower. Later, an integro quintic spline approach over a uniform partition was discussed in [13]. Unfortunately, the method in [13] has two drawbacks. On the one hand, the deduce process is very complicated and the method needs seven additional boundary conditions; on the other hand, there are no interpolation error analysis for the derivatives approximation. In an earlier paper [10], the integro quartic spline was used therein. However, the computational method in [10] was also very complicated. In fact, it required us to solve $2n+2$ linear equations. At the same time, the method

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also needs four additional boundary conditions besides the integro values. Furthermore, all the error estimations were not given in [10]. To overcome the above-mentioned drawbacks and improve the results in [10–13], a new technique is desired.

In this paper, we will present a new integro quartic spline interpolation method by using quartic B-splines. Our new method has many advantages. First, our new algorithm is more concise than the methods in [10,11,13]. It only needs solving a linear system composed of $n+4$ linear equations with a five-band coefficient matrix, hence it is easy to be implemented. Second, our algorithm produces numerical results with higher accuracy than the results in [10–13]. Especially, we point out that our new integro quartic spline $s(x)$, as a lower-degree spline than the quintic spline in [13], is able to approximate $y(x)$ with $O(h^6)$ errors at the knots, and also can approximate $y''(x)$ with $O(h^4)$ errors at the knots. This shows that the new integro quartic spline possesses super convergence orders (sixth order and fourth order, respectively) in approximating function values and second-order derivative values at the knots. This is a surprise. Third, our method can work successfully even without any boundary conditions, while the method in [10,11,13] needs four, three and seven additional boundary conditions, respectively. Fourth, we also provide the derivatives approximation for $y(x)$, and the error estimations are well studied.

The remainder of this paper is organized as follows. In Section 2, we present some preliminary results of quartic B-splines. In Section 3, we give the new concise integro interpolation method by using quartic B-splines. In Section 4, we analyze the interpolation errors for the integro-interpolating quartic spline. In Section 5, we study a modified method for integro quartic spline interpolation without any boundary conditions (only involving n integro values over the subintervals), we also analyze the interpolation errors for this case, and we prove that the convergence orders are unchanged. Section 6 is devoted to numerical tests, and numerical results show that our method is very effective not only in function approximation but also in derivatives approximation. Finally, we conclude our paper in Section 7.

2. Preliminaries

For an interval $I=[a,b]$, divide it into n subintervals by the equidistant knots $x_i=a+ih$, where $\Delta_i=[x_i,x_{i+1}](i=0,1,\ldots,n-1)$ and $h=\frac{b-a}{n}$. The univariate quartic spline space over the uniform partition is defined as follows:

$$S_4(I)=\{s(x)\in C^3(I)\mid s_i(x)\in \mathbf{P_4},\ i=0,1,\ldots,n-1\},$$

where $s_i(x)$ denotes the restriction of $s(x)$ over $\Delta_i=[x_i,x_{i+1}]$, and $\mathbf{P_4}$ denotes the set of univariate quartic polynomials. $S_4(I)$ is a linear space, its dimension is $n+4$, and its elements are called quartic splines. Essentially, a quartic spline $s(x)$ is a piecewise quartic polynomial such that $s(x), s'(x), s''(x)$ and $s'''(x)$ are continuous on $[a,b]$.

Extend $I=[a,b]$ to $\widetilde{I}=[a-4h,b+4h]$ with the equidistant knots $x_i=a+ih(i=-4,-3,\ldots,n+4)$. By the results in [14,15], we obtain the explicit representations of the typical quartic B-spline $B_i(x)(i=-2,-1,\ldots,n+1)$ as follows (also see [16])

$$B_i(x)=\frac{1}{24h^4}\begin{cases}(x-x_{i-2})^4, & \text{if } x\in[x_{i-2},x_{i-1}]\\ (x-x_{i-2})^4-5(x-x_{i-1})^4, & \text{if } x\in[x_{i-1},x_i]\\ (x-x_{i-2})^4-5(x-x_{i-1})^4+10(x-x_i)^4, & \text{if } x\in[x_i,x_{i+1}]\\ (x-x_{i+3})^4-5(x-x_{i+2})^4, & \text{if } x\in[x_{i+1},x_{i+2}]\\ (x-x_{i+3})^4, & \text{if } x\in[x_{i+2},x_{i+3}]\\ 0, & \text{else}\end{cases}.$$

We list some properties of $B_i(x)$ as follows.

- $B_i(x)(i=-2,-1,\ldots,n+1)$ are linearly independent, and they form the basis splines of $S_4(I)$.
- $B_i^{(k)}(x)=B_{i+1}^{(k)}(x+h)(i=-2,-1,\ldots,n;k=0,1,2,3)$; the values of $B_i^{(k)}(x)$ at the knots are given in Table 1.
- $\sum_{i=-2}^{n+1}B_i(x)\equiv 1(x\in[a,b])$.
- $B_i(x)(i=-2,-1,\ldots,n+1)$ is non-negative and is locally supported on $[x_{i-2},x_{i+3}]$; further, we have

$$\int_{x_{i-2}}^{x_{i-1}}B_i(x)dx=\int_{x_{i+2}}^{x_{i+3}}B_i(x)dx=\frac{1}{120}h, \tag{1}$$

$$\int_{x_{i-1}}^{x_i}B_i(x)dx=\int_{x_{i+1}}^{x_{i+2}}B_i(x)dx=\frac{26}{120}h, \tag{2}$$

$$\int_{x_i}^{x_{i+1}}B_i(x)dx=\frac{66}{120}h, \tag{3}$$

$$\int_{x_j}^{x_{j+1}}B_i(x)dx=0,\quad (j\geq i+3,\ \text{or}\ j\leq i-3). \tag{4}$$