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Computing the eigenvalues of the generalized Sturm–Liouville problems based on the Lie-group $SL(2, \mathbb{R})$

Chein-Shan Liu

Department of Civil Engineering, National Taiwan University, Taipei, Taiwan

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ABSTRACT

For the generalized Sturm–Liouville problems we can construct an $SL(2, \mathbb{R})$ Lie-group shooting method to find eigenvalues. By using the closure property of the Lie-group, a one-step Lie-group transformation between the boundary values at two ends of the considered interval is established. Hence, we can theoretically derive an analytical characteristic equation to determine the eigenvalues for the generalized Sturm–Liouville problems. Because the closed-form formulas are derived to calculate the unknown left-boundary values in terms of λ, the present method provides an easy numerical implementation and has a cheap computational cost. Numerical examples are examined to show that the present $SL(2, \mathbb{R})$ Lie-group shooting method is effective.

1. Introduction

The Sturm–Liouville problem has a variety of applications in partial differential equations, vibration of continuum mechanics, and quantum mechanics. There is a continued interest in the numerical solutions of Sturm–Liouville problems with the aim to improve the convergence speed and ease of numerical implementation. In order to obtain more efficient numerical results, several numerical methods have been developed [1–11].

Although Ghelardoni et al. [12] have discussed a shooting technique for computing the eigenvalues, to our best knowledge there is no study on the Lie-group $SL(2, \mathbb{R})$ shooting method and apply it to the generalized Sturm–Liouville problem. In this paper we propose a new shooting method based on the Lie-group $SL(2, \mathbb{R})$ for computing the eigenvalues and eigenfunctions of the following *generalized Sturm–Liouville problem*:

$$\frac{d}{dx}\left[p(x)\frac{dy(x)}{dx}\right] + q(x,\lambda)y(x) = 0, \quad x_0 < x < x_f, \tag{1}$$

$$a_1(\lambda)y(x_0) + a_2(\lambda)p(x_0)y'(x_0) + a_3(\lambda)y(x_f) + a_4(\lambda)p(x_f)y'(x_f) = a_0(\lambda), \tag{2}$$

$$b_1(\lambda)y(x_0) + b_2(\lambda)p(x_0)y'(x_0) + b_3(\lambda)y(x_f) + b_4(\lambda)p(x_f)y'(x_f) = b_0(\lambda). \tag{3}$$

Here we suppose that $y^2(x_0) + [p(x_0)y'(x_0)]^2 > 0$ in order to avoid the solution of y to be zero. The present problem is that for the given functions of $p(x)$ and $q(x, \lambda)$ and all the coefficients $a_i, b_i, i = 0, \ldots, 4$ we need to calculate the eigenvalue λ and the eigenfunction $y(x)$. In the above we suppose that $p(x) > 0$, and $q(x, \lambda) > 0$ can be an arbitrary nonlinear function of x and λ. In the latter sense, Eqs. (1)–(3) constitute a *nonlinear Sturm–Liouville problem*.

The group-preserving scheme (GPS) has been developed in [13] for the integration of initial value problems (IVPs). Liu [14–16] has extended and modified the GPS for ordinary differential equations (ODEs) to solve the boundary value

E-mail address: liucs@ntu.edu.tw.

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problems (BVPs). Liu [17] could solve an inverse Sturm–Liouville problem by using the Lie-group method to find the potential function $q(x)$ with a high accuracy. In the construction of the Lie-group method for the numerical solutions of BVPs, Liu [14] has introduced the idea of one-step GPS by utilizing the closure property of the Lie-group, and hence, the new shooting method has been labeled the Lie-group shooting method (LGSM). However, this method needs to be modified for the Sturm–Liouville problem [18]. Recently, Liu [19] has developed a more powerful Lie-group adaptive method for computing a leading coefficient in the Sturm–Liouville operator by using the boundary data.

When the coefficient $q(x, \lambda)$ depends nonlinearly on the eigen-parameter λ, we have a generalized Sturm–Liouville problem. This also concerns the problems with eigen-parameter dependent boundary conditions [20–26]. The problem in Eqs. (1)–(3) essentially differs from the classical one and is more difficult to solve, and so far only a few methods have been proposed for solving the generalized Sturm–Liouville problems. Liu [26] has solved the generalized Sturm–Liouville problem by using a Lie-group shooting method to find the eigenvalues and eigenfunctions. Numerical methods for computing the eigenvalues of the generalized Sturm–Liouville problems with eigen-parameter dependent boundary conditions have been developed [21–28]. The shooting method presented in this paper is based on the Lie-group $SL(2, \mathbb{R})$, which is an extension and simpler than the previous works of Liu [18,26], which are based on the Lie-group $SO_o(2, 1)$. Here we develop a more powerful Lie-group shooting method (LGSM) directly based on the Lie-group $SL(2, \mathbb{R})$ for solving the generalized Sturm–Liouville problems. In most cases, it is not possible to obtain the eigenvalues of the generalized Sturm–Liouville problem analytically. However, in the present paper we can derive a closed-form algebraic equation to compute these eigenvalues analytically.

The remaining parts of this paper are arranged as follows. In Section 2 we propose a self-adjoint formulation of the second-order linear ODE, and develop an $SL(2, \mathbb{R})$ Lie-group shooting method for solving the boundary value problem (BVP) of second-order ODE. In Section 3, two numerical examples are given for solving the BVPs by using the newly developed $SL(2, \mathbb{R})$ Lie-group shooting method. Based on the results in Section 3, we derive a rather simple *characteristic equation* to compute the eigenvalues in Section 4. Numerical examples for the generalized Sturm–Liouville problems are given in Section 5. Finally, we draw some conclusions in Section 6.

2. An $SL(2, \mathbb{R})$ Lie-group shooting method

2.1. A group-preserving scheme (GPS)

Usually, we may write

$$u''(x) + a(x)u'(x) + b(x)u(x) = 0, \quad x > x_0 \tag{4}$$

as a system of two first-order ODEs by

$$\frac{d}{dx}\begin{bmatrix} u(x) \\ u'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b(x) & -a(x) \end{bmatrix}\begin{bmatrix} u(x) \\ u'(x) \end{bmatrix}. \tag{5}$$

However, upon letting

$$p(x) = \exp\left[\int_{x_0}^{x} a(\xi)d\xi\right] \tag{6}$$

be the *integrating factor* of Eq. (4) we have

$$\frac{d}{dx}(p(x)u'(x)) = -p(x)b(x)u(x). \tag{7}$$

It shows that the left-hand side can be managed to a total differential form by using the concept of integrating factor. Furthermore, from Eq. (7) we have a self-adjoint system:

$$\frac{d}{dx}\begin{bmatrix} u(x) \\ p(x)u'(x) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{p(x)} \\ -p(x)b(x) & 0 \end{bmatrix}\begin{bmatrix} u(x) \\ p(x)u'(x) \end{bmatrix}. \tag{8}$$

This system is better than system (5), because it allows a Lie-group symmetry of $SL(2, \mathbb{R})$:

$$\frac{d}{dx}\mathbf{G} = \mathbf{A}\mathbf{G}, \qquad \mathbf{G}(x_0) = \mathbf{I}_2, \tag{9}$$

$$\mathbf{A}(x) = \begin{bmatrix} 0 & \frac{1}{p(x)} \\ -p(x)b(x) & 0 \end{bmatrix}, \tag{10}$$

$$\det \mathbf{G} = 1. \tag{11}$$