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A parametric approach for solving a class of generalized quadratic-transformable rank-two nonconvex programs

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ABSTRACT

The aim of this paper is to propose a solution algorithm for a particular class of rank-two nonconvex programs having a polyhedral feasible region. The algorithm is based on the so-called "optimal level solutions" method. Various global optimality conditions are discussed and implemented in order to improve the efficiency of the algorithm.

1. Introduction

The aim of this paper is to study, from a theoretical, an algorithmic, and a computational point of view, the following class of rank-two nonconvex programs:

$$P: \begin{cases} \inf f(x) = \phi\left(\frac{1}{2}x^T Qx + q^T x, d^T x\right) \\ x \in X = \{x \in \Re^n : Ax \leq b\}, \end{cases} \tag{1}$$

where $A \in \Re^{m\times n}$, $b \in \Re^m$, $q, d \in \Re^n$, $Q \in \Re^{n\times n}$ is positive definite, and $X \neq \emptyset$. The scalar function $\phi(y_1, y_2)$ is assumed to be continuous and strictly increasing with respect to variable y_1, and is defined for all values in Ω, where

$$\Omega = \left\{(y_1, y_2) \in \Re^2 : \; y_1 = \frac{1}{2}x^T Qx + q^T x, \; y_2 = d^T x, \; x \in X\right\}.$$

The considered class of objective functions $f(x)$ is extremely wide, and it covers multiplicative, fractional, and d.c. quadratic functions (as it is known, a d.c. function is a function expressed by the difference of two convex functions). Just as an example, given any strictly increasing real function g_1, any positive function g_2, and any real function g_3, then the following function $f(x)$ verifies the assumptions of problem P by using $\phi(y_1, y_2) = g_1(y_1)g_2(y_2) + g_3(y_2)$ (see also [1]):

$$f(x) = g_1\left(\frac{1}{2}x^T Qx + q^T x\right) g_2\left(d^T x\right) + g_3\left(d^T x\right). \tag{2}$$

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Various particular problems belonging to this class have been studied in the literature of mathematical programming and global optimization, from both a theoretic and an applicative point of view [2–8]. For these particular problems the proposed solution algorithms are often based on branch and bound, branch and cut, or branch and reduce methods. It is worth noticing that this class covers several multiplicative, fractional, and d.c. quadratic problems (see [9–11,1,12–15]) which are used in applications such as location models, tax programming models, portfolio theory, risk theory, and data envelopment analysis (see [16,17,13,18,14,19]).

Unfortunately, the current literature does not provide any algorithm which can determine the global solution of all the problems P belonging to the class described in (1).

The aim of this paper is to propose a solution algorithm which is able to solve in an unifying approach all of the problems considered in (1) by means of the so-called "optimal level solutions" method (see [20,9,21,10,11,1,12,22,23,15,24]). It is known that this is a parametric method, which finds the optimum of the problem by determining the minima of particular subproblems. In particular, the optimal solutions of these subproblems are obtained by means of a sensitivity analysis aimed at maintaining the Karush–Kuhn–Tucker optimality conditions. Applying the optimal level solutions method to problem P, we obtain some strictly convex quadratic subproblems which are independent of function $\phi(y_1, y_2)$. In other words, different problems share the same set of optimal level solutions, and this allows us to propose a unifying method to solve all of them.

In Section 2, we describe how the optimal level solutions method can be applied to problem P; in Section 3, a solution algorithm is proposed and fully described; in Section 4 some results are proposed in order to improve the performance of the method; finally, in Section 5 the results of a deep computational test are provided and discussed, while in Section 6 some real applications are described.

2. A parametric approach

In this section, we show how problem P can be solved by means of the so-called *optimal level solutions approach* (see [10,11,1,23]). With this aim, let $\xi \in \Re$ be a real parameter, and let us define the corresponding parametrical subset of X:

$$X_\xi = \{x \in \Re^n : Ax \leqq b,\ d^T x = \xi\}.$$

In the same way, the following further subset of X can be defined:

$$X_{[\xi_1,\xi_2]} = \{x \in \Re^n : Ax \leqq b,\ \xi_1 \le d^T x \le \xi_2\}.$$

The following parametric subproblem can then be obtained just by adding to problem P the constraint $d^T x = \xi$:

$$P_\xi : \begin{cases} \min \phi\left(\frac{1}{2}x^T Qx + q^T x, \xi\right) \\ x \in X_\xi = \{x \in \Re^n : Ax \leqq b,\ d^T x = \xi\}. \end{cases}$$

The parameter ξ is said to be a *feasible level* if the set X_ξ is nonempty. An optimal solution of problem P_ξ is called an *optimal level solution*. Since $\phi(y_1, y_2)$ is strictly increasing with respect to variable y_1, for any feasible level ξ the optimal solution of problem P_ξ coincides with the optimal solution of the following strictly convex quadratic problem $\overline{P}_\xi$:

$$\overline{P}_\xi : \begin{cases} \min \frac{1}{2}x^T Qx + q^T x \\ x \in X_\xi = \{x \in \Re^n : Ax \leqq b,\ d^T x = \xi\}. \end{cases}$$

Obviously, an optimal solution of problem P is also an optimal level solution and, in particular, it is the optimal level solution with the smallest value; the idea of this approach is then to scan all the feasible levels, studying the corresponding optimal level solutions, until the minimizer of the problem is reached. Starting from an incumbent optimal level solution, this can be done by means of a sensitivity analysis on the parameter ξ, which allows us to move in the various steps through several optimal level solutions until the optimal solution is found (see [1]).

Remark 2.1. Notice that problem $\overline{P}_\xi$ admits one and only one minimum point, since its objective function is quadratic and positive definite and the feasible region X_ξ is closed. Since function $\phi(y_1, y_2)$ is strictly increasing with respect to variable y_1 and is defined for all the values in Ω, problem P_ξ admits one and only one minimum point too, the same as $\overline{P}_\xi$. As a consequence, the following logical implication holds:

$$\xi \in \Re \text{ is a feasible level} \quad \Rightarrow \quad \arg\min_{x \in X_\xi} f(x) \neq \emptyset.$$

2.1. Sensitivity analysis

Let x' be the optimal solution of problem $\overline{P}_{\xi'}$, where $d^T x' = \xi'$, and let us consider the following Karush–Kuhn–Tucker conditions for $\overline{P}_{\xi'}$: