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High-order approximation of Pearson diffusion processes

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ABSTRACT

This paper focuses on Pearson diffusions and the spectral high-order approximation of their related Fokker–Planck equations. The Pearson diffusions is a class of diffusions defined by linear drift and quadratic squared diffusion coefficient. They are widely used in the physical and chemical sciences, engineering, rheology, environmental sciences and financial mathematics. In recent years diffusion models have been studied analytically and numerically primarily through the solution of stochastic differential equations. Analytical solutions have been derived for some of the Pearson diffusions, including the Ornstein–Uhlenbeck, Cox–Ingersoll–Ross and Jacobi processes. However, analytical investigations and computations for diffusions with so-called heavy-tailed ergodic distributions are more difficult to perform. The novelty of this research is the development of an accurate and efficient numerical method to solve the Fokker–Planck equations associated with Pearson diffusions with different boundary conditions. Comparisons between the numerical predictions and available time-dependent and equilibrium analytical solutions are made. The solution of the Fokker–Planck equation is approximated using a reduced basis spectral method. The advantage of this approach is that many models for pricing options in financial mathematics cannot be expressed in terms of a stochastic partial differential equation and therefore one has to resort to solving Fokker–Planck type equations.

1. Introduction

The Fokker–Planck (or forward Kolmogorov) equation arises in a wide variety of problems of physical and biological interest including non-Newtonian flows, plasma physics, biophysics and financial mathematics. Despite all the work that has been undertaken by many scientists in this area there remain many unresolved problems associated with this equation. For example, there exists only a few exact solutions to transient problems. Beyond these, one has to resort to numerical investigations. The situation is exacerbated when the Fokker–Planck equation is nonlinear. In this paper we investigate numerical solutions of Fokker–Planck equations for so-called Pearson diffusions.

The Pearson class of diffusions is very attractive since their ergodic (stationary, equilibrium) solutions satisfy the famous Pearson equation [1]. The classification of the stationary solutions of Pearson diffusions gives six classes of probability distributions, three of them with an infinite system of moments (Gaussian, Gamma, Beta) and three others with only a finite number of moments (inverted Gamma, Student and Fisher–Snedecor). The latter are known as heavy-tailed distributions. Among the class of Pearson diffusions, the Ornstein–Uhlenbeck (OU), Cox–Ingersoll–Ross (CIR) [2], and Jacobi processes have been well studied and applied in many practical situations. However, the statistical analysis of heavy-tailed Pearson diffusions such as reciprocal gamma, Student and Fisher–Snedecor diffusions are relatively new (see [3–11], for example).

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It is now generally accepted that heavy-tailed distributions are useful for the analysis of a range of problems in application areas such as communication networks, risky assets and insurance modelling. Since for most diffusion models a closed form representation of the transition density is unavailable and its statistical analysis is often very complicated, a numerical approach becomes unavoidable.

Most effort in recent years has been expended in studying different types of Pearson diffusions using stochastic differential equations (SDEs) applying both analytical and numerical methods (see [5,12] and references therein, for example). Although stochastic methods have a number of advantages, there are still a number of issues which need further consideration and work. One of them, for example, is the issue of the computation of the non-negative diffusion term in the stochastic representation of the CIR process, which has been discussed in [13]. To elaborate on this issue further, consider the well-known CIR diffusion as SDE driven by standard Brownian motion, B_t, $t \geq 0$, that is

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dB_t, \quad t \geq 0,$$

with real parameters $a \geq 0$, $b \in \mathbb{R}$, and $\sigma > 0$. This model is widely used for modelling interest rates in financial mathematics. It is known that the strong solution is unique and preserves the non-negativity of the initial data. For the real-valued numerical implementation of the $\sqrt{X_t}$ term in stochastic simulations, the non-negativity of the numerical approximation is of paramount importance. Standard numerical schemes do not conserve the non-negativity of the solutions and overcome the problem in an ad-hoc fashion by evaluating $\sqrt{|X_t|}$ instead of $\sqrt{X_t}$ (see [14,15], for example). Indeed using usual schemes, such as Euler or Milstein, when discretising the CIR process located at 0 can lead to negative values for which the square root is not defined. There are a number of possible schemes that avoid this problem and these have been discussed in [16]. This problem associated with non-negative diffusion terms is typical for all Pearson diffusions with the exception of the OU process.

Analytical solutions have been derived for some of the Pearson diffusions, including the OU, CIR and Jacobi processes, but there has been less success for time-dependent investigations and for diffusions with heavy-tails. In this paper we present a numerical scheme for solving the associated Fokker–Planck equations which have ergodic Fisher–Snedecor, reciprocal gamma or Student distributions. In spite of the simple and tractable analytical forms of stationary (ergodic) distributions, the analytical form of time-dependent solutions are either not tractable and very complicated or even unknown. In many practical situations one has to resort to numerical techniques to determine the solution of such equations.

In this paper we utilise techniques that have been developed to perform direct numerical simulations of the Fokker–Planck equation arising from atomistic and mesoscopic models in theoretical polymer rheology [17–20]. Typically, these models involve a large number of configurational degrees of freedom. This means that all the standard methods of discretisation require a substantial number of unknowns to obtain an accurate representation of the distribution function. For this reason, initial numerical work in this area concentrated on the solution of equivalent stochastic differential equations.

Recently, however, Ammar et al. [21] devised a method for significantly reducing the number of degrees of freedom involved in the solution of the Fokker–Planck equation in high dimensions using a basis reduction method. In their paper, Ammar et al. [21] approximated the solution of the Fokker–Planck equation using a finite element basis. This technique was extended to nonlinear rheological models [22] and to transient problems [23]. High-order spectral approximations were developed in [24]. Efficiency was achieved through a dynamic construction of the basis which ensured that only basis functions containing the most representative information of the solution are retained. This technique is used in the present paper to generate high-order approximations of Pearson diffusion processes.

2. Pearson's diffusions and their classification

The study of time-dependent homogeneous diffusion processes with transition density, $p(x, t)$, and invariant distributions from the Pearson family dates from the 1930's, when Kolmogorov [25] investigated the Fokker–Planck or forward Kolmogorov equation

$$\frac{\partial}{\partial t}p(x, t) = -\frac{\partial}{\partial x}[a(x)p(x, t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[b(x)p(x, t)], \quad x \in \mathbb{R},\ t \geq 0, \tag{1}$$

with a linear drift, $a(x) = a_1x + a_0$, and a quadratic squared diffusion, $b(x) = 2d(x)$, and $d(x) = b_2x^2 + b_1x + b_0$ and observed that the invariant density $\mathfrak{p}(\cdot)$ satisfies the differential equation

$$\frac{\mathfrak{p}'(x)}{\mathfrak{p}(x)} = \frac{a(x) - d'(x)}{d(x)} = \frac{(a_1 - 2b_2)x + (a_0 - b_1)}{b_2x^2 + b_1x + b_0}, \quad x \in \mathbb{R}, \tag{2}$$

introduced in [1], in order to unify some of the most important statistical distributions.

It seems appropriate to call this important class of processes, Pearson diffusions. The class of Pearson diffusions is closed under transitions and scale transformations. Forman and Sørensen [4], for example, provide a classification of the stationary solutions of these processes (see also [5,26]).