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Weak second order S-ROCK methods for Stratonovich stochastic differential equations☆

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ABSTRACT

It is well known that the numerical solution of stiff stochastic ordinary differential equations leads to a step size reduction when explicit methods are used. This has led to a plethora of implicit or semi-implicit methods with a wide variety of stability properties. However, for stiff stochastic problems in which the eigenvalues of a drift term lie near the negative real axis, such as those arising from stochastic partial differential equations, explicit methods with extended stability regions can be very effective. In the present paper our aim is to derive explicit Runge–Kutta schemes for non-commutative Stratonovich stochastic differential equations, which are of weak order two and which have large stability regions. This will be achieved by the use of a technique in Chebyshev methods for ordinary differential equations.

1. Introduction

While it has been customary to treat the numerical solution of stiff ordinary differential equations (ODEs) by implicit methods, there is a class of explicit methods with extended stability regions that are well suited to solving stiff problems whose eigenvalues lie near the negative real axis. Such problems include parabolic partial differential equations when solved by the method of lines.

An original contribution is by van der Houwen and Sommeijer [1] who have constructed explicit s-stage Runge–Kutta (RK) methods whose stability functions are shifted Chebyshev polynomials $T_s(1 + z/s^2)$. These have stability regions along the negative real axis $[-2s^2, 0]$. The corresponding RK methods satisfy a three term recurrence relation which make them efficient to implement, but their drawback is that they are of order one. Lebedev [2,3] and Medovikov [4] have constructed high order methods by computing the zeros of the optimal stability polynomials for maximal stability. But, the method is sensitive to the ordering of these zeros and there is no recurrence relation.

Abdulle and Medovikov [5] have developed a new strategy to construct second order Chebyshev methods with nearly optimal stability regions. These methods are based on a weighted orthogonal polynomial and so the numerical methods satisfy a three term recurrence relation. In this case the stability interval is $[-l_s, 0]$ where $l_s \approx 0.82s^2$. These ideas have been extended in [6] who constructed a family of s-stage damped Chebyshev methods of order four that possess nearly optimal stability regions along the negative real axis and a three term recurrence relation. For these methods, $l_s \approx 0.35s^2$.

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One of the drawbacks with Chebyshev methods is that the stability region can collapse to $s-1$ single points on the negative real axis due to the mini–max property of Chebyshev polynomials. Accordingly, we require the modulus of the stability polynomial to be bounded by a damping factor $\eta < 1$. The stability interval shrinks slightly but a strip around the negative real axis is included in the stability region. With $\eta = 0.95$, $l_s \approx 0.81s^2$ for the second order Chebyshev methods.

In the case of stochastic differential equations (SDEs) the issues are much more complicated. Nevertheless, Abdulle and Cirilli [7] have developed a family of explicit stochastic orthogonal Runge–Kutta Chebyshev (SROCK) methods with extended mean square (MS) stability regions. These methods are of weak order one for non-commutative Stratonovich SDEs. They reduce to the first order Chebyshev methods when there is no noise term. Such an approach is important because there are very few good numerical methods for solving stiff SDEs.

We are concerned with weak second order stochastic Runge–Kutta (SRK) methods, especially derivative-free ones, for non-commutative SDEs. Kloeden and Platen [8, pp. 486–487] have proposed a derivative-free scheme of weak order two for non-commutative Itô SDEs. Tocino and Vigo-Aguiar [9] have also proposed it as an example in their RK family. Komori [10] has proposed a different scheme for non-commutative Stratonovich SDEs, which has an advantage that it can reduce the random variables that need to be simulated. This scheme, however, still has a drawback in that its computational costs linearly depend on the dimension of the Wiener process for each diffusion coefficient. Rößler [11] and Debrabant and Rößler [12] have proposed new schemes which overcome the drawback while keeping the advantage for Stratonovich or Itô SDEs. Komori and Burrage [13] have also proposed an efficient SRK scheme which overcomes the drawback by improving the scheme in [10].

Abdulle and Cirilli's approach is important because it is difficult to construct implicit or drift-implicit methods of weak order two for stiff SDEs [8,14,15]. In the present paper we shall put all these ideas together. We will construct a family of s-stage SRK methods of weak order two for non-commutative Stratonovich SDEs and with extended mean square stability regions. The methods will reduce to the second order Chebyshev methods of Abdulle and Medovikov [5] when the noise terms are set to zero. In Section 2 we will give some background material on Chebyshev methods for ODEs. In Section 3 we will give background material on SDEs. In Section 4 we will give a framework of SRK methods, while in Section 5 we will derive our new class of methods based on the stability analysis. Section 6 will present numerical results and Section 7 our conclusions.

2. Chebyshev methods for ODEs

Consider the autonomous N-dimensional ODEs given by

$$\boldsymbol{y}'(t) = \boldsymbol{f}(\boldsymbol{y}(t)), \quad t > 0, \qquad \boldsymbol{y}(0) = \boldsymbol{y}_0. \tag{1}$$

The class of s-stage RK methods for solving (1) is

$$\boldsymbol{Y}_i = \boldsymbol{y}_n + h\sum_{j=1}^{s} a_{ij}\boldsymbol{f}(\boldsymbol{Y}_j) \quad (1 \le i \le s), \qquad \boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\sum_{j=1}^{s} b_j\boldsymbol{f}(\boldsymbol{Y}_j). \tag{2}$$

For an equidistant grid point $t_n \stackrel{\text{def}}{=} nh$ $(n = 1, 2, \ldots, M)$ with step size h (M is a natural number), $\boldsymbol{y}_n$ denotes a discrete approximation to the solution $\boldsymbol{y}(t_n)$ of (1). An RK method is explicit if $a_{ij} = 0$ $(i \le j)$.

Denote by A an $s \times s$ matrix $[a_{ij}]$ and define $\boldsymbol{b} \stackrel{\text{def}}{=} [b_1\ b_2\ \cdots\ b_s]^\top$ and $\boldsymbol{e} \stackrel{\text{def}}{=} [1\ 1\ \cdots\ 1]^\top$. When we apply (2) to the linear, scalar test problem

$$y'(t) = \lambda y(t), \quad t > 0, \qquad \Re(\lambda) \le 0, \quad y(0) = y_0, \tag{3}$$

we have $y_{n+1} = R(h\lambda)y_n$ where

$$R(z) \stackrel{\text{def}}{=} 1 + z\boldsymbol{b}^\top (I - Az)^{-1}\boldsymbol{e}. \tag{4}$$

Here R is called the stability function and for explicit methods $R(z)$ is a polynomial of degree s at most, namely

$$R(z) = 1 + \sum_{j=1}^{s} z^j \boldsymbol{b}^\top A^{j-1}\boldsymbol{e}. \tag{5}$$

The stability region of (2) is $S \stackrel{\text{def}}{=} \{z \mid |R(z)| \le 1\}$. A method whose stability domain contains the whole left half of the complex plane is said to be A-stable, but such methods are by necessity implicit.

Van der Houwen and Sommeijer [1] constructed RK methods of order one that have maximal stability regions along the negative real axis, namely $[-2s^2, 0]$. These methods have a stability polynomial given by

$$R(z) = T_s(1 + z/s^2), \tag{6}$$

where $T_j(x)$ is the Chebyshev polynomial of degree j defined by $T_j(\cos\theta) \stackrel{\text{def}}{=} \cos(j\theta)$ or by the three term recurrence relation

$$T_0(x) \stackrel{\text{def}}{=} 1, \qquad T_1(x) \stackrel{\text{def}}{=} x, \qquad T_j(x) \stackrel{\text{def}}{=} 2xT_{j-1}(x) - T_{j-2}(x), \quad j \ge 2.$$