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A unified Pythagorean hodograph approach to the medial axis transform and offset approximation

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ABSTRACT

Algorithms based on Pythagorean hodographs (PH) in the Euclidean plane and in Minkowski space share common goals, the main one being rationality of offsets of planar domains. However, only separate interpolation techniques based on these curves can be found in the literature. It was recently revealed that rational PH curves in the Euclidean plane and in Minkowski space are very closely related. In this paper, we continue the discussion of the interplay between spatial MPH curves and their associated planar PH curves from the point of view of Hermite interpolation. On the basis of this approach we design a new, simple interpolation algorithm. The main advantage of the unifying method presented lies in the fact that it uses, after only some simple additional computations, an arbitrary algorithm for interpolation using planar PH curves also for interpolation using spatial MPH curves. We present the functionality of our method for G^1 Hermite data; however, one could also obtain higher order algorithms.

1. Introduction

Curve and surface offsets are geometric objects that are frequently used in various technical applications, e.g. numerically controlled machining and computer-aided manufacturing. Due to their wide applicability, studying classical (and also general) offsets of hypersurfaces has recently become an active and popular research area. Many interesting problems related to this topic have arisen, including those of the analysis of geometric and algebraic properties of offsets, determining the number and type of offset components and constructing rational parametrisations of offsets; cf. [1–7].

Describing a tool path in NURBS form is currently a universal standard in technical applications. However, free-form NURBS do not possess rational offsets in general and thus techniques of approximation for offsets must be used, especially in connection with CAD/CAM systems. Since offset approximation and trimming is usually performed at the expense of great computational effort, it is worthwhile to investigate suitable exact techniques and to study curves and surfaces with exact rational offsets. This approach led to the definition of Pythagorean hodograph (PH) curves in [8]. These special curves can be used for formulating efficient approximation and interpolation techniques for free-form shapes. Comparing methods based on PH curves to the classical approximation, not the offset but the base curve is approximated. This guarantees that all corresponding offset curves are rational and mutually equidistant, and only one approximation step is required even if more than one offset is needed.

Later, the concept of polynomial planar PH curves was generalized to space PH curves [9–13], to rational PH curves [14,15] and to the so called Pythagorean normal vector (PN) surfaces [14,1,16,17]. For a survey of shapes with Pythagorean normals (i.e., possessing rational offsets), see [18]. However, even though these shapes admit rational offsets, the usually most

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demanding part of the construction process is trimming. In practice, not the whole offset but only some of its parts are used.

As observed in [19], using the medial axis transform (MAT) representation makes the trimming procedure of inner offsets considerably simpler – only those parts of the MAT where the corresponding circle radius is less than the offset distance have to be trimmed. This gives a strong justification for studying approximation and interpolation techniques based on the so called Minkowski Pythagorean hodograph (MPH) curves. Polynomial MPH curves were defined in [20] and later generalized to rational MPH curves in [21]. Indeed, if a part of the medial axis transform of a planar domain is an MPH curve, then the corresponding domain boundary segments and all their offsets possess rational parametrisations. Interpolation and approximation methods based on MPH curves were thoroughly investigated in e.g. [22–27].

Although algorithms based on Pythagorean hodographs in the Euclidean plane and in Minkowski space share common goals, the main one being rationality of offsets of planar domains, there exist many efficient but separate techniques for Hermite interpolation based on PH and MPH curves. This shortcoming motivates the search for a unifying computational approach. The main advantage of the method presented in this paper lies in the fact that it directly uses, after only some simple completing operations, an arbitrary algorithm for interpolation using planar PH curves also for interpolation using spatial MPH curves. All details of our approach are discussed in the following sections. At this point we only reveal that this technique is based on the close interplay between spatial MPH curves and associated planar PH curves studied in [21].

The remainder of this paper is organized as follows. Section 2 recalls some basic facts concerning Euclidean and Minkowski Pythagorean hodograph curves, medial axis transforms and envelopes of one-parameter families of circles. Section 3 is devoted to a novel interpolation method with MPH curves based on planar PH splines. In this section, we formulate and analyse an algorithm for G^1 Hermite interpolation via MPH curves. The algorithm is then demonstrated on several examples in Section 4. Finally, we conclude the paper in Section 5.

2. Preliminaries

We briefly review fundamentals of rational curves with Pythagorean hodographs in the Euclidean plane and in Minkowski space and recall their close interplay. The reader is referred to [18,21] for more details.

2.1. Rational curves with rational offsets in the Euclidean plane

Consider a C^1 parametric curve $\mathbf{x}(t) = (x_1(t), x_2(t))^\top$. The *δ-offset* of $\mathbf{x}(t)$ is the set of all points in $\mathbb{R}^2$ that lie at a distance δ from $\mathbf{x}(t)$. The two branches of the offset are given by

$$\mathbf{x}_\delta(t) = \mathbf{x}(t) \pm \delta\mathbf{n}(t), \qquad \mathbf{n}(t) = \frac{\mathbf{x}'(t)^\perp}{\|\mathbf{x}'(t)\|}, \tag{1}$$

where $\|\cdot\|$ denotes the usual Euclidean norm and $\mathbf{x}'(t)^\perp = (x_2'(t), -x_1'(t))^\top$ is the rotation of $\mathbf{x}'(t)$ about the origin by the angle $-\frac{\pi}{2}$.

A study of offset rationality led to the class of planar *Pythagorean hodograph* (PH) curves (i.e., curves with rational offsets) introduced in [8]. Rational PH curves are defined as rational curves $\mathbf{x}(t) = (x_1(t), x_2(t))^\top$ fulfilling the (Euclidean) *PH condition*

$$\mathbf{x}'(t)\cdot\mathbf{x}'(t) = x_1'(t)^2 + x_2'(t)^2 = \sigma(t)^2, \tag{2}$$

where $\sigma(t) \in \mathbb{R}(t)$ is a rational function and '·' denotes the standard Euclidean inner product. In order to avoid working with piecewise representations, we consider only curves for which $\sigma(t) > 0$ in the interval of interest for the remainder of the paper. Then, $\sigma(t)$ will be called the *speed* of $\mathbf{x}(t)$.

A parametric representation of all planar rational PH curves can be obtained from their dual representation

$$(2kl : k^2 - l^2 : -g) \tag{3}$$

in the form (cf. [14,15])

$$x_1 = \frac{2(ll' - kk')g + (k^2 - l^2)g'}{2(k^2 + l^2)(kl' - k'l)}, \qquad x_2 = \frac{(k'l + kl')g - klg'}{(k^2 + l^2)(kl' - k'l)}, \tag{4}$$

where $k(t), l(t)$ are relatively prime polynomials and $g(t) = e(t)/f(t)$ is a rational function.

Pythagorean hodograph curves were originally introduced in [8] as *planar polynomial* curves. These can be readily obtained from the general formula (4) by setting

$$e(t) = 2kl\int(k^2 - l^2)m\,\mathrm{d}t - (k^2 - l^2)\int 2klm\,\mathrm{d}t, \qquad f(t) = \text{constant}, \tag{5}$$

where $m(t)$ is an arbitrary polynomial; see [15]. Consequently, we arrive at

$$x_1' = m(k^2 - l^2), \qquad x_2' = 2mkl, \qquad \sigma = m(k^2 + l^2), \tag{6}$$

which describes all polynomial solutions of (2), i.e., all polynomial PH curves in $\mathbb{R}^2$.