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A posteriori error estimates for *hp* finite element solutions of convex optimal control problems

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ABSTRACT

In this paper, we present a posteriori error analysis for *hp* finite element approximation of convex optimal control problems. We derive a new quasi-interpolation operator of Clément type and a new quasi-interpolation operator of Scott–Zhang type that preserves homogeneous boundary condition. The Scott–Zhang type quasi-interpolation is suitable for an application in bounding the errors in L^2-norm. Then *hp* a posteriori error estimators are obtained for the coupled state and control approximations. Such estimators can be used to construct reliable adaptive finite elements for the control problems.

1. Introduction

Optimal control problems have been an important topic in science and engineering. They have various application backgrounds in the operation of physical, social, and economic processes. With its wide range of applications in science and engineering, there are a lot of numerical methods for these problems, especially finite element methods. There are many theoretical and numerical studies about finite element methods for the optimal control problems. For the optimal control problems, many researchers have obtained a priori error estimates (see, e.g., [1–3]) for the standard finite element methods and (see, e.g., [4,5,3]) for the mixed finite element methods. In recent years, adaptive finite element methods are of great practical importance, and have been extensively investigated in the literature. At the heart of any adaptive finite element method is an a posteriori error indicator. Many a posteriori error estimates are obtained for the standard finite element methods (see, e.g., [6,7,3]) and for the mixed finite element methods (see, e.g., [8,3]).

To the best of our knowledge, there are many *h*-versions of adaptive finite element methods for optimal control problems (see, e.g., [9,3]). There have been many theoretical studies about *hp* finite element method (see, e.g., [10–17]). The spectral element method has been extended to optimal control problems (see, e.g., [18,19]). In fact, comparable literature for high-order element such as *hp*-version of finite element method (*hp*-FEM) for optimal control problems is rather few. In [19], the *hp* spectral element method is applied in the analysis of the optimal control problems, but it needs

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stronger condition on the control satisfying u_{hp} as defined below should be H^1-function. In this paper, we will provide *hp* a posteriori error estimates for both the state and the control with weaker condition satisfying $u_{hp} \in L^2(\Omega_U)$ as defined below only. And we also present two types of quasi-interpolation results for H^2-functions: we first exhibit a quasi-interpolation of Clément type, then further we derive a quasi-interpolation operator of Scott–Zhang type that preserves the homogeneous boundary condition. The operator of Scott–Zhang type is useful to bound the errors in the L^2-norm to derive sharper estimators. In *hp* adaptation, one has the option to split an element (*h*-refinement) or to increase its approximation order (*p*-refinement). Generally speaking, a local *p*-refinement is the more efficient method on regions where the solution is smooth, while a local *h*-refinement is the strategy suitable on elements where the solution is not smooth (see [16]). We will present an a posteriori error analysis for the *hp* finite element approximation of convex optimal control problems. For the optimal control problems, we may use higher-order *hp*-FEM space for the state y and the costate p if their regularity is allowed, while we can use lower-order *hp*-FEM space for the control u due to its limited regularity.

Throughout this paper, we mainly concentrate on the following two-dimensional convex optimal control problem

$$\min_{u\in K}\{g(y)+h(u)\}, \tag{1.1}$$

subject to the state equation

$$-\operatorname{div}(A\nabla y)=f+Bu, \quad x\in\Omega, \tag{1.2}$$

with the boundary condition

$$y=0, \quad x\in\partial\Omega, \tag{1.3}$$

where g and h are given convex functionals, K is a closed convex set, and B is a continuous linear operator. More details will be specified later. In this paper we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{W^{m,q}(\Omega)}$ and the semi-norm $|\cdot|_{W^{m,q}(\Omega)}$ (see [20]). We set $W_0^{m,q}(\Omega)\equiv\{w\in W^{m,q}(\Omega): w|_{\partial\Omega}=0\}$. We denote $W^{m,2}(\Omega)(W_0^{m,2}(\Omega))$ by $H^m(\Omega)(H_0^m(\Omega))$. For $\hat{I}=(0,1)$, $\kappa\in(0,1)$ and $q\in[1,\infty)$, we equip the space $W^{\kappa,q}(\hat{I})$ with the Slobodeckij norm $\|v\|^q_{W^{\kappa,q}(\hat{I})}=\|v\|^q_{L^q(\hat{I})}+\int_{\hat{I}}\int_{\hat{I}}\frac{|v(x)-v(y)|^q}{|x-y|^{1+q\kappa}}\mathrm{d}x\mathrm{d}y$. We denote $W^{\kappa,2}(\hat{I})$ by $H^\kappa(\hat{I})$ (see [17] for more details). Additionally, c or C denotes a general positive constant independent of h_τ, p_τ, h_{τ_U}, p_{τ_U} and $\hat{p}_{\tau_U}$ that are defined below.

This paper is organized as follows: In Section 2, we describe the *hp* finite element approximation for the convex optimal control problems (1.1)–(1.3). In Section 3, we present some technical results including two new types of quasi-interpolation results for H^2-functions. In Section 4, we derive both upper bounds and lower bounds for the error estimates of the control, the state and the costate in *hp*-FEM. In the last section, we briefly discuss some possible further work.

2. *hp*-FEM approximation of optimal control problems

In this section, we shall describe the *hp* finite element discretization for the optimal control problems (1.1)–(1.3). Let Ω and Ω_U be two bounded open sets in $\mathbb{R}^2$ with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$. We also assume that Ω and Ω_U are polygonal.

We take the state space $Y=H_0^1(\Omega)$, the control space $U=L^2(\Omega_U)$ with the inner product $(\cdot,\cdot)_U$, and $H=L^2(\Omega)$, with the inner product $(\cdot,\cdot)$. Assume that g and h are convex functionals which are continuously differentiable on $H=L^2(\Omega)$ and $U=L^2(\Omega_U)$, respectively, and h is further strictly convex. Suppose that K is a closed and non-empty convex set in U, $f\in L^2(\Omega)$, B is a continuously linear operator from U to H, and

$$A(\cdot)=(a_{i,j}(\cdot))_{2\times2}\in(W^{1,\infty}(\Omega))^{2\times2},$$

such that there is a constant $c>0$ satisfying, for any vector $X\in\mathbb{R}^2$,

$$X^tAX\geq c\|X\|^2_{\mathbb{R}^2},$$

where X^t denotes the transpose of X.

Let

$$a(v,w)=\int_\Omega(A\nabla v)\cdot\nabla w, \quad \forall v,\ w\in H^1(\Omega),$$

$$(f,w)=\int_\Omega fw, \quad \forall f,\ w\in L^2(\Omega).$$

It follows from the assumptions on A that there is a constant $c>0$ such that

$$a(v,v)\geq c\|v\|^2_Y, \quad \forall v\in Y. \tag{2.1}$$

We further assume that $h(u)\to+\infty$ as $\|u\|_{L^2(\Omega_U)}\to\infty$, and the functional $g(\cdot)$ is bounded below,

$$|(g'(v)-g'(w),q)|\leq C\|v-w\|_Y\|q\|_Y \quad \forall v,w,q\in Y. \tag{2.2}$$