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Algorithms for approximating minimization problems in Hilbert spaces

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ABSTRACT

In this paper, we study the following minimization problem

$$\min_{x\in Fix(S)\cap\Omega} \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x\|^2 - h(x),$$

where B is a bounded linear operator, $\mu \geq 0$ is some constant, h is a potential function for $\bar{\gamma} f$, $Fix(T)$ is the set of fixed points of nonexpansive mapping S and Ω is the solution set of an equilibrium problem. This paper introduces two new algorithms (one implicit and one explicit) that can be used to find the solution of the above minimization problem.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. Recall that a mapping $A : C \to H$ is called *α-inverse-strongly monotone* if there exists a positive real number α such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Let $f : C \to H$ be a ζ-contraction; that is, there exists a constant $\zeta \in [0, 1)$ such that $\|f(x) - f(y)\| \leq \zeta\|x - y\|$ for all $x, y \in C$. Recall that a mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Denote the set of fixed points of S by $Fix(S)$. Let B be a strongly positive bounded linear operator on H, that is, there exists a constant $\gamma > 0$ such that

$$\langle Bx, x\rangle \geq \gamma\|x\|^2, \quad \forall x \in H.$$

Let $A : C \to H$ be a nonlinear mapping and $F : C \times C \to R$ be a bifunction. Now we concern the following equilibrium problem is to find $z \in C$ such that

$$F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

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The solution set of (1.1) is denoted by Ω. If $A = 0$, then (1.1) reduces to the following equilibrium problem of finding $z \in C$ such that

$$F(z, y) \geq 0, \quad \forall y \in C.$$

If $F = 0$, then (1.1) reduces to the variational inequality problem of finding $z \in C$ such that

$$\langle Az, y - z\rangle \geq 0, \quad \forall y \in C.$$

Equilibrium problems which were introduced by Blum and Oettli [1] in 1994 have had a great impact and influence in pure and applied sciences. It has been shown that the equilibrium problems theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization. Equilibrium problems include variational inequalities, fixed point, Nash equilibrium and game theory as special cases. The equilibrium problems and the variational inequality problems have been investigated by many authors. Please see [2–32] and the references therein. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games and others. See, e.g., [1,33–35].

For solving equilibrium problem (1.1), Moudafi [34] introduced an iterative algorithm and proved a weak convergence theorem. Further, Takahashi and Takahashi [35] introduced another iterative algorithm for finding an element of $F(S) \cap \Omega$ and they obtained a strong convergence result. Ceng and Yao [4] introduced an iterative scheme for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points of a finite family of nonexpansive mappings in a Hilbert space and obtained a strong convergence theorem. Ceng, Schaible and Yao [3] introduced an implicit iteration scheme with perturbed mapping for equilibrium problems and fixed point problems of finitely many nonexpansive mappings. Peng and Yao [11] introduced a new hybrid-extragradient method for generalized equilibrium problems and fixed point problems and variational inequality problems.

Motivated and inspired by the works in this direction in the literature, in this paper, we will study the following minimization problem

$$\min_{x \in Fix(S)\cap\Omega} \frac{\mu}{2}\langle Bx, x\rangle + \frac{1}{2}\|x\|^2 - h(x), \tag{1.2}$$

where $\mu \geq 0$ is some constant, h is a potential function for $\bar{\gamma} f$ (i.e., $h(x) = \bar{\gamma} f(x)$, for $x \in H$). This paper introduces two new algorithms (one implicit and one explicit) that can be used to find the solution of the above minimization problem.

2. Preliminaries

Let C be a nonempty closed convex subset of a real Hilbert space H. Throughout this paper, we assume that a bifunction $F : C \times C \to R$ satisfies the following conditions:

(H1) $F(x, x) = 0$ for all $x \in C$;
(H2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(H3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;
(H4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in C$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

It is well known that P_C is a nonexpansive mapping and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{2.1}$$

Moreover, P_C is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.2}$$

and

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.3}$$

for all $x \in H$ and $y \in C$.

We need the following lemmas for proving our main results.

Lemma 2.1 ([36]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to R$ be a bifunction which satisfies conditions* (H1)–(H4). *Let $r > 0$ and $x \in C$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, if $T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}$, then the following hold: