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Nonlinear scheme with high accuracy for nonlinear coupled parabolic–hyperbolic system

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ABSTRACT

A nonlinear finite difference scheme with high accuracy is studied for a class of two-dimensional nonlinear coupled parabolic–hyperbolic system. Rigorous theoretical analysis is made for the stability and convergence properties of the scheme, which shows it is unconditionally stable and convergent with second order rate for both spatial and temporal variables. In the argument of theoretical results, difficulties arising from the nonlinearity and coupling between parabolic and hyperbolic equations are overcome, by an ingenious use of the method of energy estimation and inductive hypothesis reasoning. The reasoning method here differs from those used for linear implicit schemes, and can be widely applied to the studies of stability and convergence for a variety of nonlinear schemes for nonlinear PDE problems. Numerical tests verify the results of the theoretical analysis. Particularly it is shown that the scheme is more accurate and faster than a previous two-level nonlinear scheme with first order temporal accuracy.

1. Introduction

Coupled parabolic–hyperbolic system often appears in the studies of circled fuel reactor, radiation hydrodynamics with high temperature, thermo-elasticity, magneto-elasticity and biological problems [1–7]. For some kinds of coupled systems, there are some papers [1,2,5–14] studying the existence and smoothness of their solutions.

It is necessary to solve coupled systems numerically since usually it is difficult to find their exact solutions. There have been some studies on finite element method [15,16] and finite difference method (FDM) for coupled parabolic–hyperbolic system. In [3], a FDM on a nonlinear thermo-elasticity system in one space variable was studied and its stability analysis was presented. In [17], a FDM for the same coupled system as here was studied, a three-level linear scheme was provided, and its unconditional stability and second order convergence in both spatial and temporal variants were proved.

However, for some transient physical models, linear schemes are usually not efficient enough to depict sharp change in quite short time. Nonlinear schemes are more popular to give more precise numerical solutions. For example, a nonlinear scheme was studied in [18] for a two-temperature radiative diffusion problem, and it behaved more accurate than linear scheme with comparable costs. In [19], a nonlinear finite element approximation was studied for a coupled thermal problem expressed by elliptic system. In [20], it was pointed out that for traditional operator time-splitting methods, certain restriction on time step was needed due to stability requirement; while for nonlinear schemes, such restriction can be canceled, which means larger time steps are permissible without stability loss. Due to this merit, nonlinear schemes without operator-splitting are fitter to solve coupled systems.

In [21], a nonlinear finite difference scheme was presented for a nonlinear coupled parabolic–hyperbolic system, which was a two-level scheme, and its solution had second order spatial accuracy and first order temporal accuracy. In [22], a

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nonlinear scheme with second order time accuracy was studied for radiation diffusion coupled to material conduction where both equations were of parabolic type.

In this paper, we consider a new nonlinear finite difference scheme for the coupled parabolic–hyperbolic system. It has second order accuracy both in space and time, and has some desirable properties such as less computational cost and ease of being implemented into codes for practical applications.

Consider the nonlinear coupled parabolic–hyperbolic system as follows:

$$\begin{aligned}
&u_t - \nabla\cdot(A(X,t,u,v)\nabla u) = f(X,t,u,v,u_x,u_y,v_x,v_y), \quad X\in\Omega,\ t\in J.\\
&v_{tt} - \nabla\cdot(B(X,t,u,v)\nabla v) = g(X,t,u,v,u_x,u_y,v_x,v_y,u_t,v_t), \quad X\in\Omega,\ t\in J.\\
&u(X,t)=0, \qquad v(X,t)=0, \quad X\in\partial\Omega,\ t\in J.\\
&u(X,0)=u_0(X), \qquad v(X,0)=v_0(X), \qquad v_t(X,0)=v_{t0}(X), \quad X\in\Omega,
\end{aligned} \tag{1.1}$$

where $X=(x,y)$, $\Omega=(0,L_1)\times(0,L_2)$ is an open rectangular domain in R^2 with boundary $\partial\Omega$. $J=(0,T]$, T is a positive constant. $A, B, f, g, u_0, v_0, v_{t0}$ are known functions. Here and below, consider (1.1) with the following assumptions:

(1) There exist positive constants A_*, A^*, B_*, B^*, such that $A_*\le A(X,t,\phi,\psi)\le A^*$, $B_*\le B(X,t,\phi,\psi)\le B^*$, $\forall X\in\bar{\Omega}$, $t\in\bar{J}$, $\phi\in R$, $\psi\in R$.

(2) The partial derivatives A_t, B_t are bounded; A_u, A_v, B_u, B_v are continuous, and their derivatives with respect to t are bounded; $f_u, f_v, f_{u_x}, f_{u_y}, f_{v_x}, f_{v_y}$ and $g_u, g_v, g_{u_x}, g_{u_y}, g_{v_x}, g_{v_y}, g_{u_t}, g_{v_t}$ are bounded.

(3) Problem (1.1) is uniquely solvable, and its solution $u, v\in C^4(\bar{\Omega}\times\bar{J})$.

To get an accurate and fast simulation for coupled system (1.1), we will construct and analyze a new three-level nonlinear scheme with higher accuracy than the two-level scheme in [21], i.e., second order convergence both in space and time. An additional motivation for this study is that its convergence and stability properties are needed in the research of iteration methods for solving it. A robust iterative algorithm is necessary to fulfill fast and accurate numerical solution of nonlinear schemes (e.g., see [23,24]). We will focus on efficient iteration methods in a future paper.

We will perform rigorous theoretical analysis to demonstrate the stability and convergence properties of the new nonlinear scheme, which shows it is unconditionally stable and convergent with second order rate in both space and time. In the argument of theoretical results, difficulties arising from the nonlinearity and coupling between parabolic and hyperbolic equations are overcome, by an ingenious use of the method of energy estimation and inductive hypothesis reasoning. The reasoning method here differs from those used for linear schemes, and can be broadly applied to the studies of stability and convergence for a variety of nonlinear schemes for nonlinear PDEs. Some numerical tests are presented to validate the results of the theoretical analysis and demonstrate the new scheme is more accurate and efficient than that in [21].

The paper is organized as follows. First, an equivalent coupled system of equations, and some symbols and identities are introduced in Section 2. Next, the new nonlinear scheme is proposed in Section 3. Then, theoretical analysis on the error estimate and stability of the scheme is presented, its second order H^1 and L^2-norm spatial and temporal approximation is obtained. Convergence analysis is set forth in Section 4, and stability is discussed in Section 5, since in the reasoning procedure of stability, the conclusion of convergence analysis is needed. Then in Section 6, numerical tests are provided to confirm the results of the theoretical analysis. Finally, conclusion and generalization are given in Section 7.

2. Some symbols, notations and identities

To start with the designing of the nonlinear scheme, system (1.1) is rewritten as the following equivalent system composed of three equations with a new variant $w=v_t$ introduced.

$$\begin{aligned}
&u_t - \nabla\cdot(A(X,t,u,v)\nabla u) = f(X,t,u,v,u_x,u_y,v_x,v_y),\\
&w_t - \nabla\cdot(B(X,t,u,v)\nabla v) = g(X,t,u,v,u_x,u_y,v_x,v_y,u_t,w),\\
&v_t = w, \quad X\in\Omega,\ t\in J.\\
&u(X,t)=0, \qquad v(X,t)=0, \qquad w(X,t)=0, \quad X\in\partial\Omega,\ t\in J.\\
&u(X,0)=u_0(X), \qquad v(X,0)=v_0(X), \qquad w(X,0)=v_{t0}(X), \quad X\in\Omega.
\end{aligned} \tag{2.1}$$

Some symbols and notations are introduced to the construction and analysis of the nonlinear scheme.

Divide intervals $[0,L_1]$, $[0,L_2]$ and $[0,T]$ into J_1, J_2 and M small uniform intervals respectively. Denote $\tau=\frac{T}{M}$, $\tau_n=n\tau$, and $h_1=\frac{L_1}{J_1}$, $h_2=\frac{L_2}{J_2}$, $h=\max\{h_1,h_2\}$, $x_i=ih_1$, $y_j=jh_2$, $x_{ij}=(x_i,y_j)$. For functions ϕ,ψ, denote $\phi_{ij}=\phi(x_{ij})$, $\psi^n=\psi(\tau_n)$, $d_t\psi^{n+1}=\frac{1}{\tau}(\psi^{n+1}-\psi^n)$, $\partial_t\psi^n=\frac{1}{2\tau}(\psi^{n+1}-\psi^{n-1})$, $\partial_{tt}\psi^n=\frac{1}{\tau^2}(\psi^{n+1}-2\psi^n+\psi^{n-1})$, $\phi_{i+\frac12,j}=\frac12(\phi_{ij}+\phi_{i+1,j})$, $\phi_{i,j+\frac12}=\frac12(\phi_{ij}+\phi_{i,j+1})$, $\delta_x\phi_{i+\frac12,j}=\frac{1}{h_1}(\phi_{i+1,j}-\phi_{ij})$, $\delta_y\phi_{i,j+\frac12}=\frac{1}{h_2}(\phi_{i,j+1}-\phi_{ij})$, $\partial_x\phi_{ij}=\frac{1}{2h_1}(\phi_{i+1,j}-\phi_{i-1,j})$ and $\partial_y\phi_{ij}=\frac{1}{2h_2}(\phi_{i,j+1}-\phi_{i,j-1})$. For functions ϕ,ψ and $\Theta=A,B$, denote

$$\Theta^n_{i+\frac12,j}(\phi,\psi)=\Theta(x_{i+\frac12,j},\tau_n,\phi^n_{i+\frac12,j},\psi^n_{i+\frac12,j}), \tag{2.2}$$

$$\Theta^n_{i,j+\frac12}(\phi,\psi)=\Theta(x_{i,j+\frac12},\tau_n,\phi^n_{i,j+\frac12},\psi^n_{i,j+\frac12}), \tag{2.3}$$

$$f^n_{ij}(\phi,\psi)=f(x_{ij},\tau_n,\phi^n_{ij},\psi^n_{ij},\partial_x\phi^n_{ij},\partial_y\phi^n_{ij},\partial_x\psi^n_{ij},\partial_y\psi^n_{ij}),$$