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Spatial approximation of stochastic convolutions

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ABSTRACT

We study linear stochastic evolution partial differential equations driven by additive noise. We present a general and flexible framework for representing the infinite dimensional Wiener process, which drives the equation. Since the eigenfunctions and eigenvalues of the covariance operator of the process are usually not available for computations, we propose an expansion in an arbitrary frame. We show how to obtain error estimates when the truncated expansion is used in the equation. For the stochastic heat and wave equations, we combine the truncated expansion with a standard finite element method and derive a priori bounds for the mean square error. Specializing the frame to biorthogonal wavelets in one variable, we show how the hierarchical structure, support and cancelation properties of the primal and dual bases lead to near sparsity and can be used to simplify the simulation of the noise and its update when new terms are added to the expansion.

1. Introduction

We study linear stochastic evolution problems of the form

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + B\mathrm{d}W(t), \quad t > 0; \qquad X(0) = 0, \tag{1.1}$$

where $X(t)$ is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in a separable Hilbert space H. The operator A is the infinitesimal generator of a strongly continuous semigroup e^{tA} of bounded linear operators on H, $W(t)$ is a Q-Wiener process on a Hilbert space U, and $B : U \to H$ is a bounded linear operator. The covariance operator Q of $W(t)$ is a self-adjoint, positive semidefinite, bounded linear operator on U.

Under appropriate assumptions, (1.1) has a unique weak solution which is given by the stochastic convolution (see Section 3.2),

$$X(t) = W_A(t) := \int_0^t \mathrm{e}^{(t-s)A} B\mathrm{d}W(s).$$

The motivation for studying the stochastic convolution W_A is that this is the first step towards studying more general evolution problems driven by additive noise of the form

$$\mathrm{d}X(t) = (AX(t) + f(X(t)))\mathrm{d}t + B\mathrm{d}W(t), \quad t > 0; \qquad X(0) = X_0.$$

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This can be given a rigorous meaning as an integral equation,

$$\begin{aligned} X(t) &= e^{tA}X_0 + \int_0^t e^{(t-s)A} f(X(s))\,ds + \int_0^t e^{(t-s)A} B\,dW(s) \\ &= Y(t) + W_A(t), \end{aligned}$$

where Y satisfies

$$Y'(t) = AY(t) + f(Y(t) + W_A(t)), \quad t > 0; \qquad Y(0) = X_0.$$

Thus, once W_A is known, we may study Y by means of methods for evolution differential equations with random data. This abstract framework is sufficiently general to include the stochastic heat equation, the stochastic wave equation, and the stochastic Cahn–Hilliard equation. The above program; that is, splitting the solution of a semilinear problem into the stochastic convolution and the solution of a random PDE, is carried out, for example, for the stochastic Cahn–Hilliard equation in [1–3]. The analysis methods for W_A and Y are usually quite different, both on the PDE level and on the numerical level, and the present work is focused on the numerical approximation of the stochastic convolution W_A.

The Q-Wiener process is often represented as an orthogonal series,

$$W(t) = \sum_{k=1}^{\infty} \gamma_k^{1/2} \beta_k(t) f_k,$$

where $\{\gamma_k\}_{k=1}^{\infty}$ are the eigenvalues and $\{f_k\}_{k=1}^{\infty}$ an orthonormal basis of eigenvectors of the covariance operator Q and $\{\beta_k\}_{k=1}^{\infty}$ are independent real-valued Brownian motions. However, these eigenvectors are not always available for computations. We therefore propose an expansion in terms of an arbitrary frame which is not related to Q.

Let thus $\{\phi_j\}_{j\in\mathcal{J}}$, with countable index set $\mathcal{J}$, be a frame for U with corresponding dual frame $\{\tilde{\phi}_j\}_{j\in\mathcal{J}}$, so that $\langle \phi_j, \tilde{\phi}_j \rangle = \delta_{ij}$ and

$$f = \sum_{j\in\mathcal{J}} \langle f, \tilde{\phi}_j \rangle \phi_j, \quad f \in U,$$

see [4]. Let $J \subset \mathcal{J}$ be a finite set and define a projector P_J by

$$P_J f := \sum_{j\in J} \langle f, \tilde{\phi}_j \rangle \phi_j, \quad f \in U.$$

Define the truncated finite dimensional process

$$W^J(t) := \sum_{j\in J} \langle W(t), \tilde{\phi}_j \rangle \phi_j, \quad t \geq 0,$$

and the corresponding stochastic convolution

$$W_A^J(t) := \int_0^t e^{(t-s)A} B\,dW^J(s).$$

In Theorem 3.2 we prove a formula for the mean square of the truncation error,

$$\mathbf{E}(\|W_A(t) - W_A^J(t)\|^2) = \int_0^t \|e^{sA} B(I - P_J) Q^{1/2}\|_{\mathrm{HS}}^2\,ds,$$

which is the basis for our further analysis. Here, $\|T\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm of a bounded linear operator $T: U \to H$ given by

$$\|T\|_{\mathrm{HS}}^2 = \sum_{k=1}^{\infty} \|Tf_k\|^2 \tag{1.2}$$

for some and, hence, for any orthonormal basis $\{f_k\}_{k=1}^{\infty}$ of U.

In Section 4 we introduce the deterministic heat and wave equations and their spatial approximation by a standard Galerkin finite element method. In particular, we consider the elliptic operator $\Lambda u = -\nabla \cdot (a\nabla u) + cu$ in a spatial domain $\mathcal{D}$ with boundary condition $u = 0$ on $\partial\mathcal{D}$ as an unbounded linear operator on the Hilbert space $H = L_2(\mathcal{D})$. Its finite element approximation is denoted Λ_h.

The stochastic heat equation is then of the form (1.1) with $A = -\Lambda$, $B = I$, $H = U = L_2(\mathcal{D})$ and the spatial finite element discretization leads to the truncated stochastic convolution,

$$W_{A_h}^J(t) := \int_0^t e^{(t-s)A_h} P_h P_J\,dW(s) = \int_0^t e^{-(t-s)\Lambda_h} P_h P_J\,dW(s),$$

where $A_h = -\Lambda_h$ and P_h is the orthogonal projector onto the finite element function space.