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Existence and uniqueness of nontrivial collocation solutions of implicitly linear homogeneous Volterra integral equations

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ABSTRACT

We analyse collocation methods for nonlinear homogeneous Volterra—Hammerstein integral equations with non-Lipschitz nonlinearity. We present different kinds of existence and uniqueness of nontrivial collocation solutions and give conditions for such existence and uniqueness in some cases. Finally, we illustrate these methods with an example of a collocation problem and give some examples of collocation problems that do not fit in the cases studied previously.

1. Introduction

The aim of this paper is the numerical analysis of the nonlinear homogeneous Volterra–Hammerstein integral equation (HVHIE)

$$y(t) = (\mathcal{H}y)(t) := \int_0^t K(t,s)G(y(s))\,\mathrm{d}s, \quad t \in I := [0,T], \tag{1}$$

by means of collocation methods on spaces of local polynomials. This equation has multiple applications in physics and analysis, as for example, the study of viscoelastic materials, the renewal equation, seismic response, transverse oscillations or flows of heat (see [1,2]).

Functions K and G are called *kernel* and *nonlinearity*, respectively, and we will assume that the following *general conditions* are always held, even if they are not explicitly mentioned.

- *Over* K. The kernel $K : \mathbb{R}^2 \to [0,+\infty[$ is a locally bounded function and its support is in $\{(t,s) \in \mathbb{R}^2 : 0 \le s \le t\}$.
  For every $t > 0$, the map $s \mapsto K(t,s)$ is locally integrable, and $\int_0^t K(t,s)\mathrm{d}s$ is a strictly increasing function.
- *Over* G. The nonlinearity $G : [0,+\infty[\to [0,+\infty[$ is a continuous, strictly increasing function, and $G(0) = 0$.

Note that since $G(0) = 0$, the zero function is a solution of Eq. (1), known as *trivial solution*, and so, uniqueness of solutions is no longer a desired property for Eq. (1) because we are obviously interested in nontrivial solutions. Existence and uniqueness of nontrivial solutions of Eq. (1), as well as their properties, have been deeply studied in a wide range of cases for K and G [3–8], especially in the case of convolution equations, i.e. $K(t,s) = k(t-s)$. In general, a necessary and

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sufficient condition for the existence of a nontrivial solution is the existence of a nontrivial subsolution; that is, a positive function u such that $u(t) \leq (\mathcal{H}u)(t)$. So most of the results on existence of nontrivial solutions are, indeed, characterisations of the existence of subsolutions. For instance, in [5] the next result can be found: under the *general conditions*, Eq. (1) has a nontrivial solution if and only if there is a positive integrable function $f(x)$ such that $\int_0^x \mathcal{K}(\mathcal{F}(x) - \mathcal{F}(s))\,ds \geq G^{-1}(x)$, $x \geq 0$, where $\mathcal{K}(x) := \int_0^x k(s)\,ds$ and $\mathcal{F}(x) := \int_0^x f(s)\,ds$.

It is important to note that usually, in the analysis of solutions for non-homogeneous Volterra integral equations (and their numerical approximations), most of the existence and uniqueness theorems require that a Lipschitz condition is held by the nonlinearity (with some exceptions, for instance [9]). This is not our case, since it is well known that if the nonlinearity is Lipschitz continuous, then the unique solution of (1) is the trivial one [10]. Thus, the case we are going to consider in this paper is beyond the scope of classical results of numerical analysis of non-homogeneous Volterra integral equations, in the sense that we need a non-Lipschitz nonlinearity.

Actually, there is a wide range of numerical methods available for solving integral equations (see [11] for a comprehensive survey on the subject): iterative methods, wavelet methods [12–15], generalised Runge–Kutta methods [16,17], or even Monte Carlo methods [18]. Collocation methods [10,19] have proved to be very suitable for a wide range of equations, because of their accuracy, stability and rapid convergence. In this work, we use collocation methods to solve the nonlinear HVHIE (1) written in its implicitly linear form (see below). We also give conditions for different kinds of existence and uniqueness of nontrivial collocation solutions for the corresponding collocation equations.

We organise this paper into four sections. In Section 2, we write Eq. (1) in its implicitly linear form and we describe the corresponding collocation equations; moreover, we define the concept of nontrivial collocation solution. In Section 3, we present different kinds of existence of nontrivial collocation solutions and we give conditions for their existence and uniqueness in some cases, considering convolution and nonconvolution kernels. In Section 4, we illustrate the collocation methods and their numerical convergence with an example, showing how the errors change as the collocation points vary. Moreover, we give some examples of collocation problems that do not fit in the cases studied in the paper. Finally, we present the proofs of the main results in an Appendix at the end of the paper, for the sake of readability.

2. Preliminary concepts

Let us consider the nonlinear homogeneous Volterra–Hammerstein integral equation (HVHIE) given by (1). Taking $z := G \circ y$, Eq. (1) can be written as an *implicitly linear* homogeneous Volterra integral equation (HVIE) for z:

$$z(t) = G((\mathcal{V}z)(t)) = G\left(\int_0^t K(t,s)z(s)\,ds\right), \quad t \in I, \tag{2}$$

where $\mathcal{V}$ is the linear *Volterra operator*. So, if z is a solution of (2), then $y := \mathcal{V}z$ is a solution of (1). It is known (see [20, p. 143]) that, under suitable assumptions on the nonlinearity G, there is a one-to-one correspondence between solutions of (1) and (2). Particularly, if G is injective, then $y = G^{-1} \circ z$ and hence this correspondence is given, which is granted by the *general conditions* exposed above.

2.1. Collocation problems for implicitly linear HVIEs

First, we are going to introduce the collocation problem associated with Eq. (2) and give the equations for determining a collocation solution, that we will use for approximating a solution of (2) or (1) (see Remark 1).

Let $I_h := \{t_n : 0 = t_0 < t_1 < \cdots < t_N = T\}$ be a mesh (not necessarily uniform) on the interval $I = [0, T]$ and set $\sigma_n := \,]t_n, t_{n+1}]$ with lengths $h_n := t_{n+1} - t_n (n = 0, \ldots, N-1)$. The quantity $h := \max\{h_n : 0 \leq n \leq N-1\}$ is called the *stepsize*.

Given a set of m *collocation parameters* $\{c_i : 0 \leq c_1 < \cdots < c_m \leq 1\}$, the *collocation points* are given by $t_{n,i} := t_n + c_i h_n (n = 0, \ldots, N-1)(i = 1, \ldots, m)$ and the set of collocation points is denoted by X_h.

All this defines a *collocation problem* for Eq. (2) (see [21], [10, p. 117]), and a *collocation solution* z_h is given by the *collocation equation*

$$z_h(t) = G\left(\int_0^t K(t,s)z_h(s)\,ds\right), \quad t \in X_h, \tag{3}$$

where z_h is in the space of piecewise polynomials of degree less than m (see [10, p. 85]). Note that the identically zero function is always a collocation solution, since $G(0) = 0$.

Remark 1. From now on, a "collocation problem" or a "collocation solution" will always be referred to the implicitly linear equation (2). So, if we want to obtain an estimation of a solution of the nonlinear HVHIE (1), then we have to consider $y_h := \mathcal{V}z_h$.

As is stated in [10], a collocation solution z_h is completely determined by the coefficients $Z_{n,i} := z_h(t_{n,i}) (n = 0, \ldots, N-1)$ $(i = 1, \ldots, m)$, since $z_h(t_n + vh_n) = \sum_{j=1}^m L_j(v) Z_{n,j}$ for all $v \in \,]0, 1]$, where $L_j(v) := \prod_{k \neq j}^m \frac{v - c_k}{c_j - c_k}$ $(j = 1, \ldots, m)$