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A centerless representation of the Virasoro algebra associated with the unitary circular ensemble

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ABSTRACT

We consider the 2-dimensional Toda lattice tau functions $\tau_n(t, s; \eta, \theta)$ deforming the probabilities $\tau_n(\eta, \theta)$ that a randomly chosen matrix from the unitary group $U(n)$, for the Haar measure, has no eigenvalues within an arc (η, θ) of the unit circle. We show that these tau functions satisfy a centerless Virasoro algebra of constraints, with a boundary part in the sense of Adler, Shiota and van Moerbeke. As an application, we obtain a new derivation of a differential equation due to Tracy and Widom, satisfied by these probabilities, linking it to the Painlevé VI equation.

1. Introduction

Consider the group $U(n)$ of $n \times n$ unitary matrices, with the normalized Haar measure as a probability measure. The Weyl integral formula gives the induced density distribution on the eigenvalues of the matrices on the unit circle in the complex plane, and is given by

$$\frac{1}{n!}|\Delta_n(z)|^2 \prod_{k=1}^{n} \frac{\mathrm{d}z_k}{2\pi \mathrm{i} z_k}; \quad z_k = \mathrm{e}^{\mathrm{i}\varphi_k} \quad \text{and} \quad \Delta_n(z) = \prod_{1\le k<l\le n} (z_k - z_l).$$

Thus, for $\eta, \theta \in]-\pi, \pi[$, with $\eta \le \theta$, the probability that a randomly chosen matrix from $U(n)$ has no eigenvalues within an arc of circle $(\eta, \theta) = \{z \in S^1 | \eta < \arg(z) < \theta\}$ is given by

$$\tau_n(\eta, \theta) = \frac{1}{(2\pi)^n n!} \int_\theta^{2\pi+\eta} \cdots \int_\theta^{2\pi+\eta} \prod_{1\le k<l\le n} |\mathrm{e}^{\mathrm{i}\varphi_k} - \mathrm{e}^{\mathrm{i}\varphi_l}|^2 \mathrm{d}\varphi_1, \ldots, \mathrm{d}\varphi_n.$$

Obviously, this probability depends only on the length $\theta - \eta$. All of this is well known and we refer the reader to [1] for details. We shall denote by

$$R(\theta) = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta} \log \tau_n(-\theta, \theta), \tag{1.1}$$

the logarithmic derivative of the probability that an arc of circle of length 2θ contains no eigenvalues of a randomly chosen unitary matrix.

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The starting motivation for our work was to understand a differential equation satisfied by the function $R(\theta)$

$$R(\theta)^2 + 2\sin\theta\cos\theta R(\theta)R'(\theta) + \sin^2\theta R'(\theta)^2 = \frac{1}{2}\left(\frac{1}{4}\sin^2\theta\frac{R''(\theta)^2}{R'(\theta)} + \sin\theta\cos\theta R''(\theta) + \left(\cos^2\theta + n^2\sin^2\theta\right)R'(\theta)\right), \tag{1.2}$$

obtained in [2], from the point of view of the Adler–Shiota–van Moerbeke [3] approach, in terms of Virasoro constraints. Introducing the 2-Toda time-dependent tau functions

$$\tau_n(t,s;\eta,\theta) = \frac{1}{n!}\int_{[\theta,2\pi+\eta]^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left(e^{\sum_{j=1}^\infty (t_j z_k^j + s_j z_k^{-j})}\frac{dz_k}{2\pi i z_k}\right), \tag{1.3}$$

with $z_k = e^{i\varphi_k}$, deforming the probabilities $\tau_n(\eta,\theta) = \tau_n(0,0;\eta,\theta)$, we discover that they satisfy a set of Virasoro constraints indexed by *all* integers, decoupling into a boundary-part and a time-part

$$\frac{1}{i}\left(e^{ik\theta}\frac{\partial}{\partial\theta} + e^{ik\eta}\frac{\partial}{\partial\eta}\right)\tau_n(t,s;\eta,\theta) = L_k^{(n)}\tau_n(t,s;\eta,\theta), \quad k\in\mathbb{Z}, i=\sqrt{-1},$$

with the time-dependent operators $L_k^{(n)}$ providing a centerless representation of the *full* Virasoro algebra, see Section 2 (Theorem 2.2) for a precise statement and the proof of the result.

In their study of Painlevé equations satisfied (as functions of x) by integrals of Gessel's type $E_{U(n)}e^{x\,\mathrm{tr}(M+\overline{M})}$, where the expectation $E_{U(n)}$ refers to integration with respect to the Haar measure over the whole of $U(n)$, Adler and van Moerbeke [4] found the sl_2 subalgebra corresponding to $k=-1,0,1$, without boundary terms. The appearance of boundary terms and of a *full* centerless Virasoro algebra is to the best of our knowledge new. From this result, it is easy to obtain Eq. (1.2), using the algorithmic method of [3]. Finally, similarly to a result by the first author and Semengue [5] on the Jacobi polynomial ensemble, we show that $R(\theta)$ is the restriction to the unit circle of a function $r(z)$ defined in the complex plane, so that $\sigma(z) = -i(z-1)r(z) - n^2z/4$ satisfies a special case of the Okamoto–Jimbo–Miwa form of the Painlevé VI equation. This will be explained in Section 3 of the paper.

2. A centerless representation of the Virasoro algebra

The proof of the Virasoro constraints satisfied by the integral (1.3) is a non-trivial adaptation of the self-similarity argument exploited in the case of the Gaussian ensembles, based on the invariance of the integrals with respect to translations, see [6] and references therein. Here, we replace translations by appropriate rotations. More precisely, setting

$$dI_n(t,s,z) = |\Delta_n(z)|^2\prod_{\alpha=1}^n\left(e^{\sum_{j=1}^\infty (t_j z_\alpha^j + s_j z_\alpha^{-j})}\frac{dz_\alpha}{2\pi i z_\alpha}\right), \tag{2.1}$$

with $z_\alpha = e^{i\varphi_\alpha}$ and $|\Delta_n(z)|^2 = \prod_{1\le\alpha<\beta\le n}|z_\alpha - z_\beta|^2$, we have the fundamental next proposition.

Proposition 2.1. *The following variational formulas hold*

$$\frac{d}{d\varepsilon}dI_n\left(z_\alpha \mapsto z_\alpha e^{\varepsilon(z_\alpha^k - z_\alpha^{-k})}\right)\Big|_{\varepsilon=0} = \left(L_k^{(n)} - L_{-k}^{(n)}\right)dI_n, \tag{2.2}$$

$$\frac{d}{d\varepsilon}dI_n\left(z_\alpha \mapsto z_\alpha e^{i\varepsilon(z_\alpha^k + z_\alpha^{-k})}\right)\Big|_{\varepsilon=0} = i\left(L_k^{(n)} + L_{-k}^{(n)}\right)dI_n, \tag{2.3}$$

for all $k\ge 0$, *with*

$$L_k^{(n)} = \sum_{j=1}^{k-1}\frac{\partial^2}{\partial t_j\partial t_{k-j}} + n\frac{\partial}{\partial t_k} + \sum_{j=1}^\infty jt_j\frac{\partial}{\partial t_{j+k}} - \sum_{j=k+1}^\infty js_j\frac{\partial}{\partial s_{j-k}} - \sum_{j=1}^{k-1}js_j\frac{\partial}{\partial t_{k-j}} - nks_k, \quad k\ge 1, \tag{2.4}$$

$$L_0^{(n)} = \sum_{j=1}^\infty jt_j\frac{\partial}{\partial t_j} - \sum_{j=1}^\infty js_j\frac{\partial}{\partial s_j}, \tag{2.5}$$

$$L_{-k}^{(n)} = -\sum_{j=1}^{k-1}\frac{\partial^2}{\partial s_j\partial s_{k-j}} - n\frac{\partial}{\partial s_k} - \sum_{j=1}^\infty js_j\frac{\partial}{\partial s_{j+k}} + \sum_{j=k+1}^\infty jt_j\frac{\partial}{\partial t_{j-k}} + \sum_{j=1}^{k-1}jt_j\frac{\partial}{\partial s_{k-j}} + nkt_k, \quad k\ge 1. \tag{2.6}$$