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Estimates for the asymptotic convergence factor of two intervals

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ABSTRACT

Let E be the union of two real intervals not containing zero. Then $L_n^r(E)$ denotes the supremum norm of that polynomial P_n of degree less than or equal to n, which is minimal with respect to the supremum norm provided that $P_n(0) = 1$. It is well known that the limit $\kappa(E) := \lim_{n\to\infty} \sqrt[n]{L_n^r(E)}$ exists, where $\kappa(E)$ is called the asymptotic convergence factor, since it plays a crucial role for certain iterative methods solving large-scale matrix problems. The factor $\kappa(E)$ can be expressed with the help of Jacobi's elliptic and theta functions, where this representation is very involved. In this paper, we give precise upper and lower bounds for $\kappa(E)$ in terms of elementary functions of the endpoints of E.

1. Introduction

For $n \in \mathbb{N}$, let $\mathbb{P}_n$ denote the set of all polynomials of degree at most n with real coefficients. Let E be the union of two real intervals, i.e.

$$E := [a_1, a_2] \cup [a_3, a_4], \quad a_1 < a_2 < a_3 < a_4, \tag{1}$$

and let the supremum norm $\|\cdot\|_E$ associated with E be defined by

$$\|P_n\|_E := \max_{x\in E} |P_n(x)| \tag{2}$$

for any polynomial $P_n \in \mathbb{P}_n$. Consider the following two classical approximation problems:

$$L_n(E) := \|T_n(\cdot, E)\|_E := \min\big\{\|P_n\|_E : P_n \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}, P_n \text{ monic polynomial}\big\} \tag{3}$$

and, $0 \notin E$,

$$L_n^r(E, 0) := \|R_n(\cdot, E, 0)\|_E := \min\big\{\|P_n\|_E : P_n \in \mathbb{P}_n, P_n(0) = 1\big\}. \tag{4}$$

The optimal (monic) polynomial $T_n(x, E) = x^n + \cdots \in \mathbb{P}_n \setminus \mathbb{P}_{n-1}$ in (3) is called the Chebyshev polynomial on E and $L_n(E)$ is called the minimum deviation of $T_n(\cdot, E)$ on E. It is well known that the limit

$$\operatorname{cap} E := \lim_{n\to\infty} \sqrt[n]{L_n(E)} \tag{5}$$

exists, where $\operatorname{cap} E$ is called the Chebyshev constant or the logarithmic capacity of E. Concerning the general properties of $\operatorname{cap} C$, $C \subset \mathbb{C}$ compact, we refer to [1] and [2, Chapter 5].

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The optimal polynomial $R_n(\cdot, E, 0) \in \mathbb{P}_n$ in (4) is called the *minimal residual polynomial* for the degree n on E and the quantity $L_n^r(E, 0)$ is called the minimum deviation of $R_n(\cdot, E, 0)$ on E. Note that we say *for* the degree n but not *of* degree n since the minimal residual polynomial for the degree n on E is a polynomial of degree n or $n-1$, see [3]. As above, the limit

$$\kappa(E, 0) := \lim_{n\to\infty} \sqrt[n]{L_n^r(E, 0)} \tag{6}$$

exists, see, e.g. [4] or [5], where $\kappa(E, 0)$ is usually called the *estimated asymptotic convergence factor*. The approximation problem (4) and the convergence factor (6) arise for instance in the context of solving large-scale matrix problems by Krylov subspace iterations. There is an enormous literature on these subject, hence we would like to mention only three references, the review of Driscoll et al. [5], the book of Fischer [6] and the review of Kuijlaars [4].

In the case of two intervals, both terms, $\kappa(E, 0)$ and cap E, can be expressed with the help of Jacobi's elliptic and theta functions and this characterization goes back to the work of Achieser [7]. Since, in both cases, the representation is very involved, it is desirable to have at least estimates of a simpler form. For cap E, such estimates are given in [8–10]. In this paper, we will give a precise upper and lower bound for $\kappa(E, 0)$ in terms of elementary functions of the endpoints a_1, a_2, a_3, a_4 of E.

The paper is organized as follows. In Section 2, we recall the representations of $\kappa(E, 0)$ and cap E with the help of Jacobi's elliptic and theta functions. Using an inequality between a Jacobian theta function and the Jacobian elliptic functions, proved in Section 6, we obtain an upper and a lower bound for $\kappa(E, 0)$ in Section 3, which is the main result of the paper. In Section 4, the following extremum problem is solved: Given the length of the two intervals and the length of the gap between the two intervals, for which set of two intervals the convergence factor $\kappa(E, 0)$ gets minimal? In Section 5, as a byproduct, a new and simple lower bound for cap E is derived. Finally, in Section 6, the notion of Jacobi's elliptic and theta functions is recapitulated and several new inequalities, needed in Sections 3 and 4, are proved.

2. Representation of the asymptotic factor and the logarithmic capacity in terms of Jacobi's elliptic functions

Let E be given as in (1) such that $0 \notin E$. It is convenient to use the linear transformation

$$\ell(x) := \frac{2x - a_1 - a_4}{a_4 - a_1}, \tag{7}$$

which maps the set E onto the normed set

$$\hat{E} := [-1, \alpha] \cup [\beta, 1], \tag{8}$$

where $\alpha := \ell(a_2)$ and $\beta := \ell(a_3)$. For the corresponding Chebyshev polynomials, we have

$$T_n(x, E) = \left(\frac{a_4 - a_1}{2}\right)^n T_n(\ell(x), \hat{E}), \tag{9}$$

thus

$$L_n(E) = \left(\frac{a_4 - a_1}{2}\right)^n L_n(\hat{E}) \tag{10}$$

and

$$\operatorname{cap} E = \frac{a_4 - a_1}{2} \operatorname{cap} \hat{E}. \tag{11}$$

Concerning the minimal residual polynomial, there is

$$R_n(x, E, 0) = R_n(\ell(x), \hat{E}, \xi), \tag{12}$$

where $\xi := \ell(0)$, thus

$$L_n^r(E, 0) = L_n^r(\hat{E}, \xi) \tag{13}$$

and

$$\kappa(E, 0) = \kappa(\hat{E}, \xi), \tag{14}$$

for details, see [6, Sec. 3.2].

Let $\hat{E}$ be given as in (8) with $-1 < \alpha < \beta < 1$ and let $\xi \in \mathbb{R} \setminus \hat{E}$. Then there exists a (uniquely determined) Green's function for $\hat{E}^c := \overline{\mathbb{C}} \setminus \hat{E}$ (where $\overline{\mathbb{C}} := \mathbb{C} \cup \infty$) with pole at infinity, denoted by $g(z; \hat{E}^c, \infty)$. The Green's function is defined by the following three properties:

- $g(z; \hat{E}^c, \infty)$ is harmonic in $\hat{E}^c$.
- $g(z; \hat{E}^c, \infty) - \log |z|$ is harmonic in a neighbourhood of infinity.
- $g(z; \hat{E}^c, \infty) \to 0$ as $z \to \hat{E}, z \in \hat{E}^c$.