Journal of Computational and Applied Mathematics 236 (2011) 39–48

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On Taylor series and Kapteyn series of the first and second type

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ARTICLE INFO

MSC:
42C10
30B10
33C10

Keywords:
Kapteyn series
Taylor series
Bessel functions

ABSTRACT

We study the relation between the coefficients of Taylor series and Kapteyn series representing the same function. We compute explicit formulas for expressing one in terms of the other and give examples to illustrate our method.

1. Introduction

Series of the form [1]

$$\sum_{n=0}^{\infty} \alpha_n^{\nu} \mathrm{J}_{n+\nu}\left[(n+\nu)z\right], \tag{1}$$

and

$$\sum_{n=0}^{\infty} \alpha_n^{\mu,\nu} \mathrm{J}_{\mu+n}\left[(\mu+\nu+2n)\,z\right] \mathrm{J}_{\nu+n}\left[(\mu+\nu+2n)\,z\right], \tag{2}$$

where $\mu, \nu \in \mathbb{C}$ and $\mathrm{J}_n(\cdot)$ is the Bessel function of the first kind, are called *Kapteyn series of the first kind* and *Kapteyn series of the second kind* respectively.

Kapteyn series have a long history, going back to Lagrange's 1771 paper *Sur le Problème de Képler* [2], where he solved Kepler's equation [3]

$$M = E - \varepsilon \sin(E), \tag{3}$$

using his method for solving implicit equations [4] (now called *Lagrange inversion theorem*) and obtained [5]

$$E(M) = M + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}M^{n-1}} \sin^n(M).$$

Here M is the mean anomaly (a parameterization of time) and E is the eccentric anomaly (an angular parameter) of a body orbiting on an ellipse with eccentricity ε.

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0377-0427/$ – see front matter
doi:10.1016/j.cam.2011.03.007

In 1819 Friedrich Wilhelm Bessel published his paper *Analytische Auflösung der Kepler'schen Aufgabe* [6], where he approached (3) using a different method. First of all he observed that the function $g(M) = E(M) - M$ defined implicitly by $g = \varepsilon \sin(g + M)$ is 2π-periodic and satisfies $g(0) = 0 = g(\pi)$. Hence, $g(M)$ can be expanded in a Fourier sine series

$$g(M) = \sum_{n=1}^{\infty} b_n \sin(nM),$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} g(M) \sin(nM) \mathrm{d}M \\ &= -\frac{2}{\pi} \left[g(M) \frac{\cos(nM)}{n} \right]_0^{\pi} + \frac{2}{\pi n} \int_0^{\pi} \cos(nM) \mathrm{d}g \\ &= \frac{2}{\pi n} \int_0^{\pi} \cos(nM) d\,(E - M) \\ &= \frac{2}{\pi n} \int_0^{\pi} \cos\left[n\,(E - \varepsilon \sin E)\right] \mathrm{d}E - \frac{2}{\pi n} \int_0^{\pi} \cos(nM) \mathrm{d}M \end{aligned}$$

and hence

$$b_n = \frac{2}{\pi n} \int_0^{\pi} \cos\,(nE - n\varepsilon \sin E)\, \mathrm{d}E.$$

He then introduced the function $\mathrm{J}_n(z)$ defined by

$$\mathrm{J}_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos\,(nE - z \sin E)\, \mathrm{d}E, \quad n \in \mathbb{Z} \tag{4}$$

which now bears his name and obtained

$$E(M) = M + \sum_{n=1}^{\infty} \frac{2}{n} \mathrm{J}_n(n\varepsilon) \sin(nM). \tag{5}$$

Bessel's work on (4) was continued by other researchers including Lommel, who defined the Bessel function of the first kind by [7]

$$\mathrm{J}_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left(\frac{z}{2}\right)^{\nu+2n}, \quad \nu \in \mathbb{C}, \tag{6}$$

where $\Gamma(\cdot)$ is the Gamma function.

In 1817, Carlini [8] found an expression for the true anomaly υ (an angular parameter), defined in terms of E and ε by

$$\tan\left(\frac{\upsilon}{2}\right) = \sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \tan\left(\frac{E}{2}\right).$$

Carlini's expression reads [9]

$$\upsilon = M + \sum_{n=1}^{\infty} B_n \sin(nM),$$

where

$$B_n = \frac{2}{n} \mathrm{J}_n(n\varepsilon) + \sum_{m=0}^{\infty} \alpha^m \left[\mathrm{J}_{n-m}(n\varepsilon) + \mathrm{J}_{n+m}(n\varepsilon)\right],$$

with $\varepsilon = \frac{2\alpha}{1+\alpha^2}$. The problem considered by Carlini was to determine the asymptotic behavior of the coefficients B_n for large values of n [10]. The astronomer Johann Franz Encke drew Carl Gustav Jacob Jacobi's attention to the work of Carlini. In 1849, Jacobi published a paper improving and correcting Carlini's article [11] and in 1850 Jacobi published a translation from Italian into German [12], with critical comments and extensions of Carlini's investigation.

Bessel's research on series of the type (5) was continued in [13] in his papers [14,15] and by Kapteyn (not to be confused with his brother Jacobus Cornelius Kapteyn [16]) in the articles [17,18]. Most of the early work on Kapteyn series, together with their own results, can be found in the books in [19, Chapter XXII] and [1, Chapter 17].