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d-orthogonality of Little q-Laguerre type polynomials

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ABSTRACT

In this paper, we solve a characterization problem in the context of the d-orthogonality. That allows us, on one hand, to provide a q-analog for the d-orthogonal polynomials of Laguerre type introduced by the first author and Douak, and on the other hand, to derive new L_q-classical d-orthogonal polynomials generalizing the Little q-Laguerre polynomials. Various properties of the resulting basic hypergeometric polynomials are singled out. For $d = 1$, we obtain a characterization theorem involving, as far as we know, new L_q-classical orthogonal polynomials, for which we give the recurrence relation and the difference equation.

1. Introduction

Over the past few years, many works appeared and dealt with the notion of multiple orthogonality [1–11] which is connected with the study of vector Padé approximants, simultaneous Padé approximants, and other problems such as vectorial continued fractions, polynomials solutions of the higher order differential equations (see, for instance, [4,1,12–14]). A convenient framework to discuss examples of multiple orthogonal polynomials consists in considering a subclass of multiple orthogonal polynomials known as d-orthogonal polynomials (see, for instance, [15–35]). To draw up our contribution in this direction, studying further examples, we recall the following notations and definitions.

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its algebraic dual. We denote by $\langle u, f\rangle$ the effect of the functional $u \in \mathcal{P}'$ on the polynomial $f \in \mathcal{P}$. A polynomial sequence $\{P_n\}_{n\geq 0}$ is called a *polynomial set* (PS, for shorter) if and only if $\deg P_n = n$ for all non-negative integers n.

Definition 1.1 (*Van Iseghem [30] and Maroni [31]*)**.** Let d be a positive integer. A PS$\{P_n\}_{n\geq 0}$ is called *d-orthogonal* (d-OPS, for short) with respect to the d-dimensional vector of functionals $\Gamma = {}^t(\Gamma_0, \Gamma_1, \ldots, \Gamma_{d-1})$ if it satisfies the following orthogonality relations:

$$\begin{cases} \langle \Gamma_k, P_r P_n\rangle = 0, & r > nd + k,\ n \in \mathbb{N} = \{0, 1, 2, \ldots\}, \\ \langle \Gamma_k, P_n P_{nd+k}\rangle \neq 0, & n \in \mathbb{N}, \end{cases}$$

for each integer k belonging to $\mathbb{N}_d = \{0, 1, \ldots, d-1\}$.

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In this context, we have the following result.

Theorem 1.2 (*Maroni [31]*). *Let d be a positive integer. A PS$\{P_n\}_{n\geq 0}$ is d-orthogonal if and only if it satisfies a $(d+1)$-order recurrence relation of the type:*

$$xP_n(x) = \sum_{k=0}^{d+1} \alpha_{k,d}(n) P_{n-d+k}(x),$$

where $\alpha_{d+1,d}(n)\alpha_{0,d}(n) \neq 0$, $n \geq d$, and by convention, $P_{-n} = 0,\ n \geq 1$.

This result, for $d = 1$, is reduced to the so-called Favard Theorem [36].

The d-orthogonal polynomials of Laguerre type (d-Laguerre polynomials, for short) were introduced by the first author and Douak [19,20]:

$$\ell_n^{\vec{\alpha}_d}(x) = \ell_n^{(\alpha_1,\ldots,\alpha_d)}(x) = {}_1F_d\left(\begin{matrix}-n\\ \alpha_1+1,\ldots,\alpha_d+1\end{matrix}; x\right), \tag{1.1}$$

where ${}_rF_s$ is the hypergeometric series defined by [37]:

$${}_rF_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}; z\right) := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{r}(a_j)_k}{\prod_{j=1}^{s}(b_j)_k}\frac{z^k}{k!},$$

with $(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}$.

Such polynomials also appear among the solutions of a characterization problem considered by the authors [22,16]. It was shown in [19] that these polynomials are related to Konhauser polynomials [38] Gould–Hopper polynomials [39] and Bateman functions [40]. Recently, the first author and Gaied [25] used the d-Laguerre polynomials to express the components of the Gould–Hopper type polynomials. The first author and Douak [19] stated for d-Laguerre polynomials various properties concerning a differential equation of order $d+1$, a $(d+1)$-order recurrence relation, a generating function defined by means of the hyper-Bessel function, some differentiation formulas and a Koshlyakov formula involving the Meijer G-function.

The 2-Laguerre polynomials have been also studied by the first author and Douak [21] and by Van Assche and Yakubovich [9] in order to solve an open problem formulated by Prudnikov in [10].

Concerning how to obtain the d-dimensional vector of functionals for which the d-orthogonality of the d-Laguerre polynomials holds, the case $d = 2$ was treated by the first author and Douak [21] and the general case was solved by the second and the third authors [34].

The aim of this work is to proved for d-Laguerre polynomials their q-analogous. To this end, we recall first that the Little q-Laguerre polynomials are given in [41, p. 107] by:

$$P_n(x; a|q) = {}_2\Phi_1\left(\left.\begin{matrix}q^{-n}, 0\\ aq\end{matrix}\right| q; qx\right), \quad a \notin \{0, q^{-1}, \ldots\},\ 0 < q < 1, \tag{1.2}$$

where ${}_r\phi_s$ is the q-hypergeometric series defined in [41] by:

$${}_r\phi_s\left(\left.\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\right| q; z\right) := \sum_{n=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_n}{(b_1,\ldots,b_s;q)_n}(-1)^{n(1+s-r)} q^{\binom{n}{2}(1+s-r)}\frac{z^n}{(q;q)_n}, \tag{1.3}$$

with $(a_1,\ldots,a_r;q)_n := \prod_{j=1}^{r}(a_j;q)_n$, r a positive integer or 0 (interpreting an empty product as 1), $(a;q)_0 := 1$, $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$, $n > 1$, and $(a;q)_\infty := \lim_{n\mapsto+\infty}(a;q)_n$.

These polynomials are well-known in the theory of orthogonal polynomials as the q-analogs of the Laguerre ones [41]. That suggest us to consider the following characterization problem.

P: Find all d-orthogonal polynomials of type:

$$P_n(x; (b_s), (a_r)|q) = {}_{r+1}\phi_s\left(\left.\begin{matrix}q^{-n}, a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\right| q; x\right), \tag{1.4}$$

where $0 < q < 1$, r and s are positive integers or 0 (interpreting an empty product in (1.3) as 1), $\{a_i;\ i = 1,\ldots,r\}$ and $\{b_j;\ j = 1,\ldots,s\}$ are $r+s$ complex numbers independent of n and x such that $a_i, b_j \neq 1, q^{-1}, q^{-2}, \ldots$, and $a_i \neq b_j$.

For $r = s = 1$, these polynomials are reduced to the Little q-Laguerre polynomials given by (1.2).

Such a characterization takes into account the fact that PS which are obtainable from one another by a linear change of variable are assumed equivalent.

Notice by the way that, this problem, for the limiting case $(d, q) = (1, 1)$ was set and treated under different aspects by many authors who took as starting point for their characterizations one of the properties related to the polynomials given by