Journal of Computational and Applied Mathematics 236 (2012) 3256–3264

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Quasi-interpolation for linear functional data☆

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ARTICLE INFO

Article history:
Received 1 October 2011
Received in revised form 14 February 2012

MSC:
65D15
41A25
41A30
37B25

Keywords:
Quasi-interpolation
Radial basis function
Linear functional data
Numerical solution of the differential equation
Lyapunov function

ABSTRACT

Quasi-interpolation has been studied extensively in the literature. However, most studies of quasi-interpolation are usually only for discrete function values (or a finite linear combination of discrete function values). Note that in practical applications, more commonly, we can sample the linear functional data (the discrete values of the right-hand side of some differential equations) rather than the discrete function values (e.g., remote sensing, seismic data, etc). Therefore, it is more meaningful to study quasi-interpolation for the linear functional data. The main result of this paper is to propose such a quasi-interpolation scheme. Error estimate of the scheme is also given in the paper. Based on the error estimate, one can find a quasi-interpolant that provides an optimal approximation order with respect to the smoothness of the right-hand side of the differential equation. The scheme can be applied in many situations such as the numerical solution of the differential equation, construction of the Lyapunov function and so on. Respective examples are presented in the end of this paper.

1. Introduction

One of the simple meshless method for the numerical solution of a differential equation is the collocation by using radial basis function; see, e.g., [1] and the references therein. In fact, there are two approaches to this collocation method: the unsymmetric approach (also known as Kansa's method) proposed in [2,3], and the symmetric approach introduced in the context of Hermite–Birkhoff data interpolation [4]. While Kansa's method, as pointed in [5], may lead to a singular coefficients matrix and thus cannot always succeed. The symmetric approach, on the other hand, is unisolvable and the convergence result is given in [6].

However, both of these two collocation approaches possess no shape-preserving properties. This may be the reason why Giesl [7] pointed out that using the symmetric approach to construct the Lyapunov function for a dynamical system [8] would give an interpolant with unwanted nonnegative orbital derivative in some neighborhood of an equilibrium. To overcome the problem, [7] proposed a mend by the Taylor expansion near the equilibrium coupled with the symmetric approach. Nevertheless, both approaches showed in [8,7] all require solving a large-scale system of equations.

In the paper, we propose a quasi-interpolation scheme for linear functional data. This scheme can be used instead of the meshless collocation method in several situations such as the numerical solution of the differential equation, construction of the Lyapunov function and so on. Moreover, the scheme possesses the shape-preserving property and thus can be applied directly to construct the Lyapunov function without the mend proposed in [7].

☆ This work is supported by 973program-2006CB303102, SGST 09DZ 2272900, and the program of cultivating innovative students in key disciplines of Fudan University-EHH1411208.

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doi:10.1016/j.cam.2012.02.028

For simplifying the theoretical discussion, we focus on the univariate case. More precisely, we want to construct a quasi-interpolation scheme for the linear functional data $\{P(D)f(x_j)\}_{0\le j\le N}$ together with the n linear functionals $\{l_i(f)\}_{0\le i\le n-1}$. Following are some notations.

Denote D the differential operator that $Df(x) = f'(x)$. Let $P(D) = \sum_{i=0}^{n} a_i D^i$, where $a_i \in R$ with $a_n = 1$, be a differential operator of order n. Let $P(D)f(x) = g(x)$ and thus $P(D)f(x_j) = g(x_j)$ for $j = 0, \ldots, N$, where $\{x_j\}_{j=0}^{N}$ are pairwise distinct centers in a bounded interval $[a, b]$ with $x_0 = a$, $x_N = b$. Let $\{l_i\}_{0\le i\le n-1}$ be any family of linearly independent functionals such that the differential equation

$$\begin{cases} P(D)f(x) = g(x) \\ L(f) = F \end{cases} \tag{1.1}$$

is well-posed, where $L(f) = (l_0(f), \ldots, l_{n-1}(f))^T$ and F are n-dimensional vectors.

Quasi-interpolation, as a fundamental tool in approximation, has been studied intensively both in theory and in engineering applications, see, e.g., [9–17] and the references therein. With quasi-interpolation, one can get the solution directly without the need to solve any large-scale system of equations. However, quasi-interpolation is usually discussed only for discrete function values (or a finite linear combination of discrete function values, see [18], for instance). Obviously, this restricts the applications of quasi-interpolation, since in most cases of practical applications, we cannot sample the discrete function values directly. More commonly, we can sample the linear functional data rather than the discrete function values (e.g., solving differential equations numerically, linear functional data analysis, remote sensing, etc.). Therefore, to make quasi-interpolation available for more practical purposes, a quasi-interpolation scheme for the linear functional data is eagerly in demand.

Hon and Wu [19] proposed a quasi-interpolation scheme for the discrete values of the right-hand side of the stiff ordinary differential equation $\{(\varepsilon D^2 + D)f(x_j) = g(x_j)\}_{0\le j\le N}$ and $l_0(f) = f(x_0)$, $l_1(f) = f(x_N)$. This is the special case that $P(D) = \varepsilon D^2 + D$. Lanzara and Maz'ya [20] proposed an approximate Hermite quasi-interpolation scheme for the data $\{P(h\partial)f(jh)\}_{j\in\mathbb{Z}^d}$. Since the operator $P(h\partial)$ in [20] converges to an identity operator I as h tends to zero, from the approximation theory point of view, it is indeed an approximation of the quasi-interpolation for discrete function values. Furthermore, such a kind of data appears rarely in applications.

Motivated by the above discussions, we want to propose a quasi-interpolation scheme for the general case of the linear functional data.

The construction of our scheme consists of two parts: To facilitate the theoretical discussion, we first construct a quasi-interpolation scheme defined on the whole real line R. Then, for the purpose of practical applications, we improve this scheme and thus get a quasi-interpolation scheme defined on a bounded interval.

The paper is organized as follows.

In Section 2, we find a kernel function $\phi_h(x)$ and construct the quasi-interpolation scheme

$$Q_{P(D)}f(x) = h\sum_{j\in\mathbb{Z}} g(jh)P(-D_x)\phi_h(x - jh) + B(x)(L(B(\cdot)))^{-1}\left[L(f) - L\left(h\sum_{j\in\mathbb{Z}} g(jh)P(-D)\phi_h(\cdot - jh)\right)\right] \tag{1.2}$$

for the linear functional data on uniformly distributed centers over the whole real line R, i.e., $\{P(D)f(jh) = g(jh)\}_{j\in\mathbb{Z}}$ and $L(f)$. Error estimate of the scheme is presented in the following theorem

Theorem 1.1. *Let $g(x)$ be a $C^l(R)$ function with compact support and Ω be an arbitrary bounded interval of R. Then for the given linear functional data: $\{g(jh)\}_{j\in\mathbb{Z}}$ and $L(f)$, and any prescribed integer υ satisfying $1 \le \upsilon \le l$, we can find a kernel function $\phi_h(x)$ defined by formula (2.5), such that the quasi-interpolant $Q_{P(D)}f(x)$ of $f(x)$, for $x \in \Omega$, can provide an approximation order of $\mathcal{O}(h^\upsilon)$.*

In Section 3, we construct the main quasi-interpolation scheme for the linear functional data $\{g(x_j)\}_{0\le j\le N}$ and $\{l_i(f)\}_{0\le i\le n-1}$ as

$$\begin{aligned} Q_{P(D)}f(x) &= h\sum_{j=-M}^{N+M} \tilde{g}(x_j)P(-D_x)\phi_h(x - x_j) \\ &\quad + B(x)(L(B(\cdot)))^{-1}\left[L(f) - L\left(h\sum_{j=-M}^{N+M} \tilde{g}(x_j)P(-D)\phi_h(\cdot - x_j)\right)\right], \end{aligned} \tag{1.3}$$

where $\tilde{g}(x)$ is defined in Eq. (3.1). The scheme is available for applications. Moreover, the error estimate given in the above theorem is still valid.

Numerical examples of applying the scheme in numerical solution of differential equation and construction of the Lyapunov function are discussed in Section 4. Finally, conclusions and discussions are given in Section 5.