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The reproducing kernel method for solving the system of the linear Volterra integral equations with variable coefficients

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ABSTRACT

In this paper, we apply the reproducing kernel method to give the exact solution and approximate solution for the system of the linear Volterra integral equations with variable coefficients. Some examples are given, showing its effectiveness and convenience. Finally, the numerical results obtained by the reproducing kernel method are superior to those obtained by other methods in Farshid Mirzaee (2010) [4], Tahmasbi and Fard (2008) [5], Saeed and Ahmed (2008) [8].

1. Introduction

In this paper, we shall apply the reproducing kernel method to the following system of the linear Volterra integral equations with variable coefficients

$$\begin{aligned} a_{11}(x)f_1(x) - b_{11}\int_0^x k_{11}(x,t)f_1(t)dt + a_{12}(x)f_2(x) - b_{12}\int_0^x k_{12}(x,t)f_2(t)dt = u_1(x) \\ a_{21}(x)f_1(x) - b_{21}\int_0^x k_{21}(x,t)f_1(t)dt + a_{22}(x)f_2(x) - b_{22}\int_0^x k_{22}(x,t)f_2(t)dt = u_2(x) \end{aligned} \tag{1}$$

where $a_{11}(x), a_{12}(x), a_{21}(x), a_{22}(x)$ are arbitrary smooth functions defined on the interval [0, 1], $b_{11}, b_{12}, b_{21}, b_{22}$ are given constants. We assume that Eqs. (1) has a unique solution.

The Volterra integral equation arises in many physical applications, such as potential theory and Dirichlet problems, electrostatics, mathematical problems of radiative equilibrium, the particle transport problems of astrophysics and reactor theory, and radiative heat transfer problems. Many powerful mathematical methods such as the Galerkin method, collocation method, Taylor series, Legendre wavelets and recently the homotopy perturbation method, power series method, Adomain's method and others [1–8] have been proposed to obtain exact and approximate solutions for solving the linear Volterra integral equations system. The application of reproducing kernel method in linear and nonlinear problems has been developed by many researchers, because this method is easy to obtain the exact solution with the series form and get approximate solution with higher precision [9–17]. Now, this method will be used to deal with the system of linear Volterra integral equations with variable coefficients. In particular, this method can be extended to obtain an approximate solution and also an exact solution of a system of higher-order linear integral equations.

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2. The reproducing kernel method

In this paper, we shall give the exact solution and approximate solution of Eqs. (1) in the reproducing kernel space. We assume that Eqs. (1) have the unique solution.

To deal with the system, we consider Eqs. (1) as

$$\mathbf{U}(x) = \mathbf{VF}(x) \tag{2}$$

where operator

$$\begin{aligned} &\mathbf{V} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} : W[0,1] \oplus W[0,1] \to W_1[0,1] \oplus W_1[0,1] \\ &(v_{ij}f_j)(x) = a_{ij}(x)f_j(x) - b_{ij}\int_0^x k_{ij}(x,t)f_j(t)dt \quad i,j = 1,2. \end{aligned} \tag{3}$$

$$\begin{aligned} &\mathbf{U}(x) = \begin{bmatrix} u_1(x) \\ u_2(x) \end{bmatrix} \in W_1[0,1] \oplus W_1[0,1] \\ &\mathbf{F}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} \in W[0,1] \oplus W[0,1]. \end{aligned} \tag{4}$$

2.1. The spaces needed in the paper

- The inner product space $W[0,1] \oplus W[0,1]$ is defined as

  $$W[0,1] \oplus W[0,1] = \left\{ \mathbf{F}(x) = (f_1(x), f_2(x))^T \middle| f_1(x), f_2(x) \in W[0,1] \right\}.$$

  The inner product and norm are defined by

  $$\langle \mathbf{F}(x), \mathbf{G}(x) \rangle = \sum_{i=1}^{2} (f_i(x), g_i(x))_W, \quad \mathbf{F}(x), \mathbf{G}(x) \in W[0,1] \oplus W[0,1]$$

  $$\|\mathbf{F}(x)\|^2 = \sum_{i=1}^{2} \|f_i(x)\|_W^2, \quad \mathbf{F}(x) \in W[0,1]. \tag{5}$$

  It is easy to verify that $W[0,1] \oplus W[0,1]$ is a Hilbert space in the sense of the definition of inner product (5). Also, $W_1[0,1] \oplus W_1[0,1]$ is a Hilbert space in a similar manner.
- The space $W_1[0,1]$ (see [16]) is defined by
  $W_1[0,1] = \{f(x) | f(x)$ is an absolute continuous real-valued function on the interval $[0,1]$ and $f'(x) \in L^2[0,1]\}$.

  It is equipped with the inner product

  $$(f(x), g(x))_{W_1} = f(0)g(0) + \int_0^1 f'(x)g'(x)dx, \quad f(x), g(x) \in W_1$$

and the norm $\|f\|_{W_1} = \sqrt{(f,f)_{W_1}}, f(x) \in W_1$.

Theorem 2.1. *The space $W_1[0,1]$ is a reproducing kernel space with the reproducing kernel function*

$$r(x,y) = \begin{cases} 1+y, & y \le x \\ 1+x, & y > x. \end{cases} \tag{6}$$

That is, for every $x \in [0,1]$ and $f(x) \in W_1[0,1]$, it follows that

$$(f(x), r(x,y))_{W_1} = f(y).$$

- The space $W[0,1]$ (see [16]) is defined by

  $W[0,1] = \{f(x) | f'(x)$ is an absolute continuous real-valued function on the interval $[0,1]$ and $f''(x) \in L^2[0,1]\}$.
  It is equipped with the inner product

  $$(f(x), g(x))_W = f(0)g(0) + f'(0)g'(0) + \int_0^1 f''(x)g''(x)dx, f(x), g(x) \in W$$

and the norm $\|f\|_W = \sqrt{(f,f)_W}, f(x) \in W$.