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Some projection methods with the BB step sizes for variational inequalities

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ABSTRACT

Since the appearance of the Barzilai–Borwein (BB) step sizes strategy for unconstrained optimization problems, it received more and more attention of the researchers. It was applied in various fields of the nonlinear optimization problems and recently was also extended to optimization problems with bound constraints. In this paper, we further extend the BB step sizes to more general variational inequality (VI) problems, i.e., we adopt them in projection methods. Under the condition that the underlying mapping of the VI problem is strongly monotone and Lipschitz continuous and the modulus of strong monotonicity and the Lipschitz constant satisfy some further conditions, we establish the global convergence of the projection methods with BB step sizes. A series of numerical examples are presented, which demonstrate that the proposed methods are convergent under mild conditions, and are more efficient than some classical projection-like methods.

1. Introduction

Let Ω be a nonempty closed convex subset of $\mathcal{R}^n$, and F be a mapping from $\mathcal{R}^n$ into $\mathcal{R}^n$. A variational inequality problem, denoted by VI(Ω, F), consists in finding a vector $u^* \in \Omega$, such that

$$(u - u^*)^T F(u^*) \geq 0, \quad \forall u \in \Omega. \tag{1}$$

Variational inequalities have many important applications in fields such as mathematical programming, network economics, transportation research, game theory and regional sciences (see Refs. [1–6], for example).

Among the various numerical methods for solving VI problems, the projection-like methods are the simplest, especially when the projection onto the feasible set Ω is easy to implement. For example, when Ω is the nonnegative orthant, or a box, or a ball, projection methods require the lowest computational cost. The original projection method, which is usually called as Goldstein–Levitin–Polyak projection method, is introduced in [7,8]. For a given starting point $u_0 \in \mathcal{R}^n$, their method updates the iterates u_{k+1} via the following recursion:

$$u_{k+1} = P_\Omega[u_k - \beta_k F(u_k)],$$

where β_k is a judiciously chosen positive step size. Unfortunately, this method is globally convergent under some strong assumptions, i.e., if we denote L as the Lipschitz constant and η as the strongly monotone modulus, then it is convergent with the step size β_k satisfying

$$0 < \beta_L \leq \beta_k \leq \beta_U < \frac{2\eta}{L^2}.$$

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It is worth pointing out that the efficiency of this approach depends heavily on the estimations of Lipschitz constant L and the strongly monotonicity modulus η. In many cases, it might be difficult to estimate the modulus L and η even if F is an affine mapping. To overcome this difficulty, He et al. [9] introduced a self-adaptive version of the Goldstein–Levitin–Polyak projection method. Their method can find a suitable step size via a line search procedure, and the resulting algorithm improved deeply the efficiency of practical computation. Some developments in this direction are referred to [10–13] for example.

The self-adaptive strategy is attractive, since it avoids the difficult, if not impossible, task of estimating the parameters such as strong monotonicity modulus, the Lipschitz constant and so on. Moreover, the numerical results reported in [11,9, 13] indicated that the methods are more efficient than the corresponding algorithms with fixed step size. However, these advantages are achieved at the cost of large number of function evaluations and projections onto the feasible set, since it got a suitable step size via a line search procedure. In some cases, this will be time consuming, especially for large-scale problems. It is therefore natural to investigate some projection methods with 'simple' step size strategy that needs as little as possible function evaluations and projections.

In this paper, we propose such a 'simple' projection algorithm, which generates the iterative sequence $\{u_k\}$ via the following recursion:

$$u_{k+1} = P_{\Omega}[u_k - \gamma \beta_{k+1} F(u_k)] \tag{2}$$

where $\gamma > 0$ is a constant and β_{k+1} is defined by

$$\text{(BB-I)} \qquad \beta^{\mathrm{I}}_{k+1} = \frac{\|u_k - u_{k-1}\|^2}{(u_k - u_{k-1})^T [F(u_k) - F(u_{k-1})]}, \tag{3}$$

or

$$\text{(BB-II)} \qquad \beta^{\mathrm{II}}_{k+1} = \frac{(u_k - u_{k-1})^T [F(u_k) - F(u_{k-1})]}{\|F(u_k) - F(u_{k-1})\|^2}. \tag{4}$$

Note that the task of the algorithm per iteration is just a function evaluation and a projection, which makes it suitable for large-scale problems.

The above projection methods (2) with step sizes (3) or (4) are nothing else but extensions of projection methods with Barzilai–Borwein (BB) step sizes, which can be traced back to the pioneering work proposed in [14], where they were used in gradient methods for unconstrained optimization problems:

$$\min_{u \in \mathcal{R}^n} f(u), \tag{5}$$

where f is smooth and its gradient is available. The gradient method for solving (5) is an iterative method of the form

$$u_{k+1} = u_k - \alpha_k \nabla f(u_k),$$

where α_k is a stepsize defined by

$$\alpha^{\mathrm{I}}_k = \frac{s^T_{k-1} s_{k-1}}{s^T_{k-1} y_{k-1}} \quad \text{or} \quad \alpha^{\mathrm{II}}_k = \frac{s^T_{k-1} y_{k-1}}{y^T_{k-1} y_{k-1}}, \tag{6}$$

where $s_{k-1} = u_k - u_{k-1}, y_{k-1} = \nabla f(u_k) - \nabla f(u_{k-1})$ and $k \geq 2$. The step sizes (6) are called BB step sizes, and the corresponding gradient methods are BB methods.

Unfortunately, the global convergence of BB methods is still an open problem for general case (some convergence results are referred to [14–16]). However, where the god closes a door, somewhere he opens a window. A large amount of numerical results show that BB methods outperform the existing gradient-type methods. Because of its high efficiency in practical computation, some ones extended it to bound constrained optimization problems [17–19]. Moreover, it has been fruitfully used in many applications, see, e.g., [20–25].

As we know, a special situation where the VI(Ω, F) can be formulated as an constrained optimization problem is that $F(u)$ is the gradient of a differentiable function $f(u) : \mathcal{R}^n \to \mathcal{R}$, i.e., $F(u) = \nabla f(u)$, in which case the VI(Ω, F) is equivalent to solving the following convex minimization problem:

$$\min_{u \in \mathcal{R}^n} \{f(u) \mid u \in \Omega\}.$$

According to [26, Theorem 4.16], when the mapping F is differentiable, F satisfies the above condition if and only if the Jacobian matrix $\nabla F(u)$ is symmetric for all u. However, we cannot expect the symmetric condition to hold in many practical equilibrium problems. That is the another reason we extend the step sizes (6) to general VI problems. Since the step sizes (3) and (4) are similar to the BB step sizes (6), we call the proposed methods as projected BB (PBB) methods and denote them as PBB-I and PBB-II with respect to BB-I and BB-II step size, respectively. In this paper, we prove their convergence for some special case, i.e., the VI problems with strongly monotone and Lipschitz continuous mappings, and the strong monotonicity modulus and Lipschitz constant satisfy some conditions. Then, most importantly, we compare the PBB-I and PBB-II methods with some existing numerical algorithms with well-established convergence results. Our purpose is to show that, even