Journal of Computational and Applied Mathematics 236 (2011) 602–620

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Interface procedures for finite difference approximations of the advection–diffusion equation

Jing Gong [a,*], Jan Nordström [b]

[a] Department of Information Technology, Scientific Computing Division, Uppsala University, Box 337, SE-751 05 Uppsala, Sweden

[b] Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden

ARTICLE INFO

*Keywords:*
High order finite difference methods
Numerical stability
Accuracy
Interface conditions
Summation-by-parts
Weak boundary conditions

ABSTRACT

We investigate several existing interface procedures for finite difference methods applied to advection–diffusion problems. The accuracy, stiffness and reflecting properties of various interface procedures are investigated.

The analysis and numerical experiments show that there are only minor differences between various methods once a proper parameter choice has been made.



## 1. Introduction

The conventional multi-block methodology for structured meshes is often, for efficiency and ease of mesh generation, used in computational physics (see [1–7]). A stable and accurate coupling at the block interfaces is therefore of utmost importance. However, there are many potential traps and possibilities for failure. Instabilities introduced at the block boundaries or interfaces are often handled by adding artificial dissipation. When advection is the dominant transport process, excessive amounts can easily reduce the accuracy. The artificial interfaces will also inevitably introduce numerical reflections, and care must be taken to minimize them. Another third important aspect when constructing interface procedures is to minimize the potential additional stiffness due to a large spectral radius.

The development of numerical schemes that overcome the problems mentioned above is an ongoing challenge, especially for high order finite difference methods. Strictly stable and accurate high order finite difference methods for both hyperbolic, parabolic and incompletely parabolic problems were derived in [8–15]. These methods employ the so-called Summation-by-Parts (SBP) operators and the Simultaneous Approximation Term (SAT) procedure for imposing boundary conditions; see [16,8,11,17,15,18]. With well-posed boundary conditions for the continuous problem, SBP operators and the SAT procedure, it is straightforward to prove stability using the energy-method. The methods discussed above have been implemented and tested in realistic flow calculations; see [19–22].

In [8,12] various versions of the SAT method in multiple domains were presented. That work was continued in [23] where the theoretical properties of interface procedures were investigated in detail. The main focus in [23] was on the stability and formal accuracy properties of the various schemes. We continue this investigation and focus on the stiffness and reflecting properties of different interface treatments. For clarity, we follow the path in [23], and consider one-dimensional problems in this paper. However, the SAT formulation can easily be extended to several space dimensions and to complicated boundary conditions (see [12,13,24,14,19–21]). Examples of other types of hybrid methods and approaches can be found in [25–31].

In Section 2, we derive conditions for well-posedness of the continuous advection–diffusion problem. Section 3 deals with the various semi-discrete multiple domain problems. We present the formulations and give a short theoretical overview of the existing stability theory. The size and location of the eigenvalues for both the continuous and discrete problems are

* Corresponding author.
*E-mail address:* jing.gong@gmail.com (J. Gong).

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doi:10.1016/j.cam.2011.08.009

considered in Section 4. In Section 5, we perform numerical experiments and compare the different interface procedures. We present both one- and two-dimensional calculations. Conclusions are drawn in Section 6.

## 2. The continuous problem

Consider the advection–diffusion problem in one space dimension,

$$u_t + au_x = \varepsilon u_{xx} + F, \quad 0 \le x \le 1,\ t > 0, \tag{1a}$$

$$\alpha u(0,t) + \beta u_x(0,t) = g^L(t), \quad t \ge 0, \tag{1b}$$

$$\gamma u(1,t) + \delta u_x(1,t) = g^R(t), \quad t \ge 0, \tag{1c}$$

$$u(x,0) = f(x), \quad 0 \le x \le 1, \tag{1d}$$

where $a, \varepsilon > 0$ and $\varepsilon \ll a$. In most cases we use $F = 0$ and we limit ourself to Robin boundary conditions with $\beta, \delta \neq 0$. The functions $F, g^L, g^R$ and $f$ are the data of the problem.

**Remark.** When the solution can be estimated in terms of all types of data, the problem (1) is called strongly well posed, see [32] for more details.

Let the inner product for real valued functions $a, b \in L^2[0,1]$ be defined by $(a,b) = \int_0^1 ab\,\mathrm{d}x$ and the corresponding norm by $\|a\|^2 = (a,a)$. The energy method applied to (1) with $F = 0$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|^2\right) + 2\varepsilon\|u_x\|^2 = \left(a + \frac{2\alpha}{\beta}\varepsilon\right)\left(u(0,t) - \frac{\varepsilon}{\beta}\frac{1}{a + \frac{2\alpha}{\beta}\varepsilon}g^L(t)\right)^2 - \left(a + \frac{2\gamma}{\delta}\varepsilon\right)\left(u(1,t) - \frac{\varepsilon}{\delta}\frac{1}{a + \frac{2\gamma}{\delta}\varepsilon}g^R(t)\right)^2$$
$$- \left(\frac{\varepsilon}{\beta}\right)^2\left(\frac{1}{a + \frac{2\alpha}{\beta}\varepsilon}\right)g^L(t)^2 + \left(\frac{\varepsilon}{\delta}\right)^2\left(\frac{1}{a + \frac{2\gamma}{\delta}\varepsilon}\right)g^R(t)^2. \tag{2}$$

Hence an energy estimate is obtained if

$$a + \frac{2\alpha}{\beta}\varepsilon < 0 \quad \text{and} \quad a + \frac{2\gamma}{\delta}\varepsilon > 0. \tag{3}$$

**Remark.** With the choice (3), the last two terms in (2) are positive but bounded since they contain only boundary data.

We have proved the following proposition.

**Proposition 2.1.** *With condition (3) satisfied, the problem (1) is strongly well posed.*

## 3. The semi-discrete problem

In this section we give a short theoretical overview of the existing stability theory for interface procedures. Most of the material, in scattered form, can be found in [8,12,23,21,33–35] but is summarized here for completeness. Section 3.1 deals with the single domain problem and the general SBP–SAT theory while Section 3.2 deals with the specifics related to the multiple domain problem.

### *3.1. Single domain in one-dimension*

Consider the problem (1) discretized on the single domain $[0,1]$ with a uniform mesh of $(N+1)$ points. The vector $\mathbf{u} = [u_0, u_1, \ldots, u_N]$ is the discrete approximation of $u$. The discrete approximation of $u$ at the grid point $i$ is denoted $u_i$. $\mathbf{u}_x$ and $\mathbf{u}_{xx}$ are the approximations of $u_x$ and $u_{xx}$, respectively. By using the SBP operators constructed in [9,15] we have

$$\begin{aligned}
&\mathbf{u}_x = D_1\mathbf{u} = P^{-1}Q\mathbf{u},\\
&\mathbf{u}_{xx} = D_{2n}\mathbf{u} = D_1(D_1\mathbf{u}) = \left(P^{-1}Q\right)^2\mathbf{u}, \quad \text{or}\\
&\mathbf{u}_{xx} = D_{2c}\mathbf{u} = P^{-1}(-A + BS)\mathbf{u},
\end{aligned} \tag{4}$$

where $A$ is a matrix with that satisfies $A + A^T \ge 0$. $P$ is a symmetric positive definite matrix. $Q$ is an almost skew-symmetric matrix that satisfies

$$Q + Q^T = B = \mathrm{diag}([-1, 0, \ldots, 0, 1]). \tag{5}$$