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A variant of the IDR(*s*) method with the quasi-minimal residual strategy

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ABSTRACT

The IDR(*s*) method proposed by Sonneveld and van Gijzen is an effective method for solving nonsymmetric linear systems, but usually with irregular convergence behavior. In this paper, we reformulate the relations of residuals and their auxiliary vectors generated by the IDR(*s*) method in matrix form. Then, using this new formulation and motivated by other QMR-type methods, we propose a variant of the IDR(*s*) method, called QMRIDR(*s*), for overcoming the disadvantage of its irregular convergence behavior. Both fast and smooth convergence behaviors of the QMRIDR(*s*) method can be shown. Numerical experiments are reported to show the efficiency of our proposed method.

1. Introduction

In this paper, we consider the solution of large, sparse and nonsymmetric linear systems of the form

$$A\boldsymbol{x} = \boldsymbol{b} \tag{1}$$

where A is a nonsingular $n \times n$ real matrix, and $\boldsymbol{b}$ is a real vector of order n.

Over the past few decades, Krylov subspace methods for solving nonsymmetric linear systems (1) have been studied in depth since the appearance of pioneering work [1,2], such as the biconjugate gradient method (Bi-CG) [3], the generalized minimal residual method (GMRES) [4], the conjugate gradient squared method (CGS) [5], the biconjugate gradient stabilized method (Bi-CGSTAB) [6] and the generalized product-type method based on Bi-CG (GPBi-CG) [7]. There are more discussions on the Krylov subspace methods in the review papers [8–11] and books [12,13].

IDR(*s*), a class of generalized algorithms for the IDR method [14] for linear systems (1), was recently proposed by Sonneveld and van Gijzen in [15]. The IDR(*s*) method is different from traditional Krylov subspace methods, but with the relationship that IDR(1) is mathematically equivalent to Bi-CGSTAB for even IDR(1) residuals, and the IDR(*s*) method with $s > 1$ is competitive with the Krylov subspace methods. The relation between IDR and Bi-CGSTAB was discussed in [16]. IDR was explained by Gutknecht in [17]; it was viewed as a Petrov–Galerkin method and a Ritz–IDR version was given in [18]. Some variants based on the IDR method have been proposed. A new IDR(*s*) variant obtained by imposing bi-orthogonalization conditions was developed in [19]. Exploiting the merits of BiCGstab(ℓ) [20] for avoiding the potential breakdown, especially for skew-symmetric or nearly skew-symmetric systems, IDRStab and GBi-CGSTAB(s, L) were proposed with higher order stabilization polynomials in [21,22], respectively. A block version of IDR(*s*) was presented

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for solving linear systems with multiple right-hand sides in [23]. The IDR method was also used to solve eigenvalue problems in [24].

Although the IDR(s) method is different from traditional Krylov subspace methods, its convergence history for the norms of residuals also shows quite an irregular convergence behavior like the Lanczos-type product methods. In practice, a method with smoother convergence behavior is more desirable. For example, when linear solvers are used as the intermediate steps for solving the partial differential equations (PDEs), a usual way of fixing the number of iteration steps is adopted to stop the inner iterations. In this case, better approximations can be obtained by the linear solvers with smoother convergence behavior. In this paper, we reformulate the relations of residuals and their auxiliary vectors generated by the IDR(s) method in matrix form. Then, on the basis of this new formulation and motivated by the QMR-type methods [25–27], we propose a variant of the IDR(s) method, called QMRIDR(s), to overcome the disadvantage of its irregular convergence behavior. Both fast and smooth convergence behaviors of the QMRIDR(s) method are expected.

This paper is organized as follows. A brief review of the IDR(s) method is given in Section 2. In Section 3, we reformulate the relations of residuals and their auxiliary vectors generated by the IDR(s) method and discuss the derivative process of the QMRIDR(s) method. Numerical experiments are reported in Section 4. Finally, we make some concluding remarks in Section 5.

Throughout this paper, for a vector $\boldsymbol{u}$ and a matrix M, let $\boldsymbol{u}^T$ be the transpose of the vector; $\|\boldsymbol{u}\|$ always denotes the Euclidean norm $\|\boldsymbol{u}\|_2 = \sqrt{\boldsymbol{u}^T\boldsymbol{u}}$. $\boldsymbol{u}_i$ denotes its ith element and $M_{i,j}$ is the entry in the ith row and jth column. I_n represents the identity matrix of size n. We use MATLAB notation $X(:, i : j)$ to denote a submatrix of X which consists of columns i to j. We will not emphasize the size of a matrix or vector if it is apparent from the context and there is no confusion.

2. The IDR(s) method

In this section, we review the IDR(s) method. Before that some preliminary information is given.

Definition 2.1. The Krylov subspace $\mathcal{K}_k(A, \boldsymbol{y})$ of order k for the matrix A and generating vector $\boldsymbol{y}$ is defined as

$$\mathcal{K}_k(A, \boldsymbol{y}) = \text{span}\{\boldsymbol{y}, A\boldsymbol{y}, \ldots, A^{k-1}\boldsymbol{y}\}.$$

From the definition, we know that $\mathcal{K}_k(A, \boldsymbol{y}) \subseteq \mathcal{K}_{k+1}(A, \boldsymbol{y})$. Furthermore, there is a value v, called the grade of $\boldsymbol{y}$, satisfying $\mathcal{K}_{v-1}(A, \boldsymbol{y}) \subset \mathcal{K}_v(A, \boldsymbol{y}) = \mathcal{K}_{v+1}(A, \boldsymbol{y}) = \cdots$, which implies that subspace $\mathcal{K}_k(A, \boldsymbol{y})$ is invariant for $k \geq v$.

Let $\boldsymbol{x}_0$ be an initial approximation of systems (1) and $\boldsymbol{r}_0$ be the corresponding residual $\boldsymbol{b} - A\boldsymbol{x}_0$. Suppose that $\mathcal{G}_0 = \mathcal{K}_v(A, \boldsymbol{r}_0)$ with v being the grade of $\boldsymbol{r}_0$ and let $\mathcal{S}$ be a subspace in $\mathbb{R}^n$. Define a sequence of subspaces $\mathcal{G}_j$ by recursion as

$$\mathcal{G}_j = (I - \omega_j A)(\mathcal{G}_{j-1} \cap \mathcal{S})$$

in which the ω_j's are nonzero parameters.

Under the assumption that subspace $\mathcal{S} \cap \mathcal{G}_0$ does not contain a nontrivial invariant subspace of A, the following result of the IDR theorem [15,14] is obtained: $\mathcal{G}_j \subset \mathcal{G}_{j-1}$, i.e., $\mathcal{G}_j$ is a proper subset of $\mathcal{G}_{j-1}$. This fact implies that the sequence of nested subspaces $\mathcal{G}_j$ is finite and $\mathcal{G}_j = \{\boldsymbol{0}\}$ for some $j \leq v$.

On the basis of the theorem, the IDR(s) method was proposed in [15]. When the $s + 1$ residuals $\boldsymbol{r}_{k-s}, \ldots, \boldsymbol{r}_k$ in $\mathcal{G}_{j-1}$ have been constructed, the idea of the IDR(s) method is to construct the next $s + 1$ residuals $\boldsymbol{r}_{k+1}, \ldots, \boldsymbol{r}_{k+s+1}$ in $\mathcal{G}_j$. The construction process continues with the proper choice of the parameter ω_j, and approximations $\boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_{k+s+1}$ are extracted simultaneously. Finally, when the residual is constructed in $\{\boldsymbol{0}\}$, the exact solution will be obtained. The main derivative process of the IDR(s) method is shown as follows.

Let P be a prescribed matrix with the order of $n \times s$. The null space of the transpose of P is set as the subspace $\mathcal{S}$, i.e., $\mathcal{S} = \mathcal{N}(P^T)$. The suggestion of orthogonalizing a set of random vectors for P was made in [15].

Suppose that $\Delta\boldsymbol{r}_i = \boldsymbol{r}_{i+1} - \boldsymbol{r}_i$; an auxiliary vector $\boldsymbol{v}_k$ is defined as

$$\boldsymbol{v}_k = \boldsymbol{r}_k - \sum_{l=1}^{s} \gamma_l \Delta\boldsymbol{r}_{k-l} \tag{2}$$

$$= (1 - \gamma_1)\boldsymbol{r}_k + \sum_{l=1}^{s-1} (\gamma_l - \gamma_{l+1})\boldsymbol{r}_{k-l} + \gamma_s \boldsymbol{r}_{k-s} \tag{3}$$

in which the γ_l's are parameters. It is obvious that $\boldsymbol{v}_k \in \mathcal{G}_{j-1}$ for any γ_l's. Moreover, $\boldsymbol{v}_k$ is also limited in $\mathcal{S}$ by the setting of parameters γ_l in the IDR(s) method. From the definition of $\mathcal{S}$, the parameters γ_l can be determined under the condition that

$$P^T\boldsymbol{v}_k = 0. \tag{4}$$