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Uniform approximation of min/max functions by smooth splines[☆]

Guohui Zhao [a,*], Zhirui Wang [a,b], Haining Mou [c]

[a] School of Mathematical Sciences, Dalian University of Technology, Dalian, 116024, Liaoning, China

[b] Consorzio RFX, Associazione Euratom ENEA sulla fusione, Corso Stati Uniti 4, Padova 35127, Italy[1]

[c] College of Mathematics and Computational Science, China University of Petroleum, Dongying, 257061, Shandong, China

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ABSTRACT

In some optimization problems, $\min(f_1, \ldots, f_n)$ usually appears as the objective or in the constraint. These optimization problems are typically non-smooth, and so are beyond the domain of smooth optimization algorithms. In this paper, we construct smooth splines to approximate $\min(f_1, \ldots, f_n)$ uniformly so that such optimization problems can be dealt with as smooth ones.

1. Introduction

Non-smooth optimization and semi-smooth optimization have been studied extensively. Typical non-smooth optimization problems are the minimax problem and problems with $\min(f_1, \ldots, f_n)$ or $\max(f_1, \ldots, f_n)$ in the constraint. Several methods have been proposed to solve such problems, such as the subgradient method [1], the cutting plane method [2], the trust region method [3], and the smoothing method [4–9]. The maximum entropy method is a typical method to smooth $\max(f_1, \ldots, f_n)$. It uses $F_p(x) = \frac{1}{p} \ln\left\{\sum_{i=1}^{n} p \cdot f_i(x)\right\}$ to smooth $\phi(x) = \max(f_1(x), \ldots, f_n(x))$. It has the properties that $0 \le F_p(x) - \phi(x) \le \ln(n)/p$ and $F_p(x)$ is decreasing with respect to p.

This paper is mainly concerned with the construction of a new smoothing method, which is a generalization of the method in [10,11] for smoothing $\min(x, 0)$ to a method for smoothing $\min(x_1, \ldots, x_n)$. We construct a smooth spline $s(x_1, \ldots, x_n)$ to approximate $\min(x_1, \ldots, x_n)$ uniformly. The composite function $s(f_1, \ldots, f_n)$ then approximates $\min(f_1, \ldots, f_n)$ uniformly. In the following sections, P_k denotes the space of polynomials of degree at most k, and $\prod v_k = v_1 \cdots v_n$ denotes the n-cone $\{v_0 + t_1\overrightarrow{v_0v_1} + t_2\overrightarrow{v_0v_2} + \cdots + t_n\overrightarrow{v_0v_n}, t_i \ge 0\}$.

2. Formulation of splines

Let us first recall the formulation of multivariate splines. Let D be a bounded polyhedral domain of R^n which is partitioned with irreducible algebraic surfaces into cells Δ_i, $i = 1, \ldots, N$. The partition is denoted by Δ. A function $f(x)$ defined on D is a spline function if $f(x) \in C^r(D)$ and $f(x)|_{\Delta_i} = p_i \in P_k$, which is expressed for short as follows: $f(x) \in S_k^r(D, \Delta)$. With the partitioning surfaces being planes, Wang obtained the following results. Let Δ_i and Δ_j be two adjacent cells with partitioning plane $p_{ij} = 0$. $f(x) \in C^r(\Delta_i \cap \Delta_j)$ if and only if $p_i - p_j = p_{ij}^{r+1} q_{ij}$, where q_{ij} is called a smooth cofactor of $f(x)$

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* Corresponding author.
E-mail addresses: ghzhao6917@yahoo.com.cn, ghzhao6961@hotmail.com (G. Zhao), wzrwork@gmail.com (Z. Wang), hainingmou@yahoo.com.cn (H. Mou).

[1] Present address.

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on the partitioning plane $p_{ij} = 0$. Further, $f(x) \in S_k^r(D, \Delta)$, if and only if there exists a smooth cofactor on each interior partitioning plane and $\sum_{p_{ij} \in L_k} p_{ij}^{r+1} q_{ij} = 0$, where L_k is the set of partitioning planes sharing a common interior line. Detailed information about multivariate splines can be found in [12,13].

3. Homogenous Morgan–Scott partition

To approximate $\min(f_1, \ldots, f_n)$ with splines, we consider the following homogenous Morgan–Scott partition for an n-cone $\sigma = \prod \nu_k$. We construct a new n-cone $\prod \omega_k$ inside σ, where $\omega_k = \alpha \sum_{j \neq k, j>0} \nu_j + (1-(n-1)\alpha)\nu_k$, $\frac{1}{n} < \alpha < \frac{1}{n-1}$. Then we partition σ into $2^n - 1$ n-cones $\pi_M = \prod_{i \in M} \omega_i \prod_{j \in N/M} \nu_j$, where $M = \{i_1, \ldots, i_m\}$ is a nonempty subset of the ordered set $N = \{1, 2, \ldots, n\}$. This partition is called the homogenous Morgan–Scott partition of type one, and is denoted by Δ_{MS}^1. If we refine it by partitioning each π_M into $m!n$-cones $\pi_M^\rho = \prod_{r=1}^m \omega_r^\rho \prod_{j \in N/M} \nu_j$, where $\omega_r^\rho = \frac{\omega_{\rho(i_1)+\rho(i_2)+\cdots+\rho(i_r)}}{r}$ and ρ runs through all the permutation of M, we obtain a new partition, which is called the homogenous Morgan–Scott partition of type two, and is denoted by Δ_{MS}^2. We have constructed an $s_2^1(R^n)$ spline $s_1(x)$ with support Δ_{MS}^1 and an $s_3^2(R^n)$ spline $s_2(x)$ with support Δ_{MS}^2 in [14,15].

Now we construct a concrete homogenous Morgan–Scott partition. Let $\nu_0 = -m(1, \ldots, 1)$, $\nu_i = (0, \ldots, 0, m, 0, \ldots, 0)$ and $\epsilon_i = \epsilon(1, \ldots, 1, 0, 1, \ldots, 1)$, with $0 < \epsilon < m$. Then we take the intersection points p_i of lines $\nu_0\epsilon_i$ with the plane $\nu_1 \cdots \nu_n$ as ω_i. The coordinates of p_i are given as follows:

$$x_i = \frac{m^2 - \epsilon m(n-1)}{(n-1)\epsilon + mn}, \qquad x_j = \frac{(2\epsilon + m)m}{(n-1)\epsilon + mn} \quad \text{for } j \neq i.$$

The plane $\prod_{r \neq i} \nu_r$ is $\sum x_r - (n+1)x_i - m = 0$. We can scale the s_2^1 homogenous spline obtained in [14] so that the polynomial on the cell $\left(\prod_{r \neq i} \nu_r\right)\omega_i$ is $\frac{1}{2m(n+1)}\left(\sum x_r - (n+1)x_i - m\right)^2$. Subtracting $\frac{1}{n+1}\sum x_i - \frac{m}{2(n+1)}$ from $s_1(x)$, and letting m approach infinity, we obtain an s_2^1 spline which approximates $\min(x_1, \ldots, x_n)$ uniformly. Similarly, we can scale the s_3^2 homogenous spline $s_2(x)$ obtained in [14,15] so that the polynomial on the cell $\left(\prod_{r \neq i} \nu_r\right)\omega_i$ is

$$-\frac{1}{3m^2(n+1)}\left(\sum x_r - (n+1)x_i - m\right)^3.$$

Adding $\frac{1}{n+1}\sum x_i - \frac{m}{3(n+1)}$ to $s_2(x)$, and letting m approach infinity, we obtain an s_3^2 spline which approximates $\min(x_1, \ldots, x_n)$ uniformly.

4. Explicit expressions of the splines obtained

First we consider the s_2^1 spline. The plane $\sum_{r=1}^n a_{i_r} x_{i_r} + d = 0$ between two cells $\omega_{i_1} \cdots \omega_{i_q} \nu_{i_{q+1}} \cdots \nu_{i_n}$ and $\omega_{i_1} \cdots \omega_{i_q} \omega_{i_n} \nu_{i_{q+1}} \cdots \nu_{i_{n-1}}$ is

$$\left(1 + \frac{\epsilon}{m}\right)(x_{i_1} + \cdots + x_{i_q}) + \left[(2-n)\frac{\epsilon}{m} - q\right] x_{i_n} + \epsilon = 0.$$

Letting m approach infinity, we then have $x_{i_1} + \cdots + x_{i_q} - q x_{i_n} + \epsilon = 0$. The limit polynomial on the cell $\omega_{i_1}\nu_{i_2}\cdots\nu_{i_n}$ is x_{i_1}. Now we construct limit polynomials on other cells by induction. According to Section 2, the conformal equation from $\omega_{i_1}\nu_{i_2}\cdots\nu_{i_n}$ to $\omega_{i_1}\omega_{i_2}\nu_{i_3}\cdots\nu_{i_n}$ and to $\omega_{i_2}\nu_{i_1}\nu_{i_3}\cdots\nu_{i_n}$ is

$$x_{i_1} + C_1(x_{i_1} - x_{i_2} + \epsilon)^2 = x_{i_2} + C_1'(x_{i_2} - x_{i_1} + \epsilon)^2.$$

That is, $C_1 = C_1' = \frac{1}{4\epsilon}$. So the limit polynomial on the cell $\omega_{i_1}\omega_{i_2}\nu_{i_3}\cdots\nu_{i_n}$ is $x_{i_1} + \frac{1}{4\epsilon}(x_{i_1} - x_{i_2} + \epsilon)^2 = x_{i_2} + \frac{1}{4\epsilon}(x_{i_2} - x_{i_1} + \epsilon)^2$.

Suppose that the limit polynomial on the cell $\omega_{i_1}\cdots\omega_{i_{q-2}}\nu_{i_{q-1}}\cdots\nu_{i_n}$ is $s_2^1(x)$, the conformal equation from $\omega_{i_1}\cdots\omega_{i_{q-2}}\nu_{i_{q-1}}\cdots\nu_{i_n}$ to $\omega_{i_1}\cdots\omega_{i_{q-1}}\nu_{i_q}\cdots\nu_{i_n}$ and to $\omega_{i_1}\cdots\omega_{i_{q-2}}\omega_{i_{q-1}}\omega_{i_q}\nu_{i_{q+1}}\cdots\nu_{i_n}$ and to $\omega_{i_1}\cdots\omega_{i_{q-2}}\omega_{i_q}\nu_{i_{q-1}}\nu_{i_{q+1}}\cdots\nu_{i_n}$ and to $\omega_{i_1}\cdots\omega_{i_{q-2}}\nu_{i_{q-1}}\cdots\nu_{i_n}$ is

$$c_{q-2}\left(\sum_{r=1}^{q-2} x_{i_r} - (q-2)x_{i_{q-1}} + \epsilon\right)^2 + c_{q-1}\left(\sum_{r=1}^{q-1} x_{i_r} - (q-1)x_{i_q} + \epsilon\right)^2$$
$$= c_{q-2}'\left(\sum_{r=1}^{q-2} x_{i_r} - (q-2)x_{i_q} + \epsilon\right)^2 + c_{q-1}'\left(\sum_{r=1}^{q-2} x_{i_r} + x_{i_q} - (q-1)x_{i_{q-1}} + \epsilon\right)^2.$$

That is, $\frac{c_{q-1}}{c_{q-2}} = \frac{q-2}{q}$ or $\frac{c_q}{c_{q-1}} = \frac{q-1}{q+1}$. So the limit polynomial on the cell $\omega_{i_1}\cdots\omega_{i_{q-2}}\omega_{i_{q-1}}\omega_{i_q}\nu_{i_{q+1}}\cdots\nu_{i_n}$ is

$$s_2^1(x) = x_{i_1} + \sum_{r=1}^{q-1} c_r \left(\sum_{l=1}^{r} x_{i_l} - r x_{i_{r+1}} + \epsilon\right)^2, \qquad c_1 = \frac{1}{4\epsilon}, \frac{c_{r+1}}{c_r} = \frac{r}{r+2}.$$