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The correspondence between multivariate spline ideals and piecewise algebraic varieties

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ABSTRACT

As a piecewise polynomial with a certain smoothness, the spline plays an important role in computational geometry. The algebraic variety is the most important subject in classical algebraic geometry. As the zero set of multivariate splines, the piecewise algebraic variety is a generalization of the algebraic variety. In this paper, the correspondence between piecewise algebraic varieties and spline ideals is discussed. Furthermore, Hilbert's Nullstellensatz for the piecewise algebraic variety is also studied.

1. Introduction

It is well known that, according to the classical Weierstrass theorem, any continuous function can be uniformly approximated by polynomials on a bounded domain. Therefore polynomials play an important role in approximation theory [1]. Unfortunately, the global property of polynomials is so strong that a polynomial can be determined solely by its properties on a neighborhood of a given point in the domain. However, splines, as piecewise polynomials, can be used to approximate any continuous, smooth, and even discontinuous function within any given tolerance [2]. Moreover, splines are easy to store, to evaluate, and to manipulate on digital computers. Splines have become fundamental tools of computational geometry, geometric modeling, numerical analysis, approximation theory, optimization, etc. [3–7,2]. Of central importance, perhaps, are univariate *B*-splines (or basic splines) first studied in some detail by Schoenberg in 1946 [8].

In 1975, Wang [5] pioneered the use of algebraic geometry in studying the theory of multivariate splines and discovered the fundamental theorem (Theorem 2.1) of multivariate splines, called the *smoothing cofactor-conformality* method. In a series of papers [9–12], Billera and Rose used the methods of homological and commutative algebra to study the algebraic properties and dimension of multivariate spline space, and the approach was further developed by Stiller, Schenck and Stillman [13–17]. Recently, Plaumann [4] studied the positivity of C^0 splines over a simplicial complex with the potential for application in optimization.

The *algebraic variety*, as the most important subject in classical algebraic geometry [18–21], is defined to be the intersections of hypersurfaces represented by multivariate polynomials. Because the objects are mainly represented by piecewise polynomials (splines), the *piecewise algebraic variety* defined as the intersection of surfaces represented by multivariate splines is a new topic in algebraic geometry and computational geometry. Moreover, studying the algebraic and geometric properties of the piecewise algebraic varieties is also important both in theory and in practice. For the recent researches on piecewise algebraic varieties, we refer the reader to [2,22–31].

The purpose of this paper is to introduce the basic algebro-geometric properties of spline ideals and piecewise algebraic varieties. In Section 2, we recall the smoothing cofactor-conformality method and some algebraic properties of multivariate splines. Next, piecewise algebraic varieties are presented in Section 3, followed by spline ideals and their properties in Section 4. Finally, the correspondence between piecewise algebraic varieties and spline ideals is studied in Section 5. Finally, we conclude the paper in Section 6.

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2. Multivariate spline space

Let $A^n := \{(a_1, \ldots, a_n) | a_i \in k, i = 1, \ldots, n\}$ be an n-dimensional affine space over an algebraically closed field k, and $D \subset A^n$ be a simply connected domain. A finite number of hyperplanes can be used to form a *partition* Δ on $D \subset A^n$. Then D is divided into a finite number of *cells*, $\delta_1, \ldots, \delta_T$. The boundary of each cell is called the *face* of Δ. Without loss of generality, we assume that D is a polyhedron in k^n. This means that Δ is a pure, hereditary polyhedral complex [32] and its cells are facets of Δ. The $(n-1)$-dimensional faces of Δ, $S_1, \ldots, S_E$, are called the *edges*. Let $l_i = 0$ be the equation of the affine hyperplane containing S_i. The zero-dimensional faces, $V_1, \ldots, V_Q$, are called *vertices* of Δ. If the face lies on the boundary of D, then it is called the *boundary face*; otherwise it is called an *interior face*.

Denote by $P_d(\Delta)$ the collection of piecewise polynomials of degree at most d

$$P_d(\Delta) := \{p | p_i = p|_{\delta_i} \in P_d, i = 1, 2, \ldots, T\},$$

where P_d is the space of n-variable polynomials of degree at most d. For an integer $0 \le \mu < d$, we say that

$$S_d^\mu(\Delta) := \{s | s \in C^\mu(D) \cap P_d(\Delta)\}$$

is a *multivariate spline space* with smoothness μ and total degree d over Δ. By using Bezout's theorem in algebraic geometry, Wang [5] discovered the following fundamental theorem on multivariate splines (for convenience, we consider bivariate spline space here).

Theorem 2.1 (*[5]*). *$s \in S_d^\mu(\Delta)$ if and only if the following conditions are satisfied:*

(1) *For each interior edge of Δ, which is defined by $S_i : l_i = 0$, there exists the so-called* smoothing cofactor *q_i such that*

$$s_{i1} - s_{i2} = l_i^{\mu+1} q_i,$$

where the polynomials s_{i1}, s_{i2} are determined by the restriction of s on the two cells δ_{i1} and δ_{i2} with S_i as the common edge and $q_i \in P_{\alpha-(\mu+1)}$, $\alpha = \max\{\deg(s_{i1}), \deg(s_{i2})\}$.

(2) *For any interior vertex V_j of Δ, the following* conformality conditions *are satisfied*

$$\sum [l_i^{(j)}]^{\mu+1} q_i^{(j)} \equiv 0,$$

where the sum runs over all the interior edges $S_i^{(j)} : l_i^{(j)} = 0$ passing through V_j, and the signs of the smoothing cofactors $q_i^{(j)}$ are fixed in such a way that when a point crosses $S_j^{(j)}$ from δ_{i2} to δ_{i1}, it goes around V_j in a counterclockwise manner.

More details about theory of multivariate splines using the smoothing cofactor-conformality method can be found in [5–7]. Billera and Rose [9–12,33] extended this method by using homological and commutative algebra to study the algebraic properties and dimension of $S_d^\mu(\Delta)$, and the approach was further developed by Stiller, Schenck and Stillman [13–17].

The space

$$S^\mu(\Delta) = \{s | s \in C^\mu(D) \cap P(\Delta)\}$$

is called the multivariate spline space with smoothness μ over Δ, where $P(\Delta)$ is the collection of piecewise polynomials over Δ. Obviously, $S_d^\mu(\Delta)$ is the subset of $s \in S^\mu(\Delta)$ such that the restriction of s to each cell in Δ is a polynomial of degree d or less. In fact, $S^\mu(\Delta)$ is a Nötherian ring [7,2]. Obviously, the polynomial ring $k[x_1, \ldots, x_n]$ is a subset of $S^\mu(\Delta)$, and it is a proper subset if partition Δ is generic.

Suppose that

$$B^\mu(\Delta) = \left\{ (g_1, \ldots, g_E) | \sum_{\delta \in c} g_\delta l_\delta^{\mu+1} = 0, \forall c \in C, g_i \in k[x_1, \ldots, x_n], i = 1, \ldots, E \right\},$$

where C denotes the set of cycles in the dual graph G_Δ of Δ. [32,33] presented the following algebraic meaning of Theorem 2.1 by considering a module $B^\mu(\Delta)$ built out of syzygies on the $l_i^{\mu+1}$.

Theorem 2.2 (*[32,33]*).

(1) *$S^\mu(\Delta)$ has the structure of a module over the ring $k[x_1, \ldots, x_n]$.*
(2) *$S_d^\mu(\Delta)$ is a finite-dimensional vector subspace of $S^\mu(\Delta)$.*
(3) *$S^\mu(\Delta)$ is isomorphic to $B^\mu(\Delta) \bigoplus k[x_1, \ldots, x_n]$ as a $k[x_1, \ldots, x_n]$-module.*
(4) *If G_Δ is a tree (i.e., a connected graph with no cycles), then $S^\mu(\Delta)$ is a free module for all $\mu \ge 0$.*

For each i, $1 \le i \le Q$, there is a unique function $X_i \in S_1^0(\Delta)$ such that

$$X_i(V_j) = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

The X_i are called *Courant functions* of Δ. The *face ring* of Δ, denoted as $k[\Delta]$, is defined as the quotient ring $k[\Delta] = k[x_1, \ldots, x_n]/I_\Delta$, where I_Δ is the ideal generated by square-free monomials not supported by faces of Δ. Billera [10] showed that in fact $S^0(\Delta)$ equals the algebra generated by the Courant functions over k.