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Pseudospectral method for quadrilaterals

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ABSTRACT

In this paper, we investigate the pseudospectral method on quadrilaterals. Some results on Legendre–Gauss-type interpolation are established, which play important roles in the pseudospectral method for partial differential equations defined on quadrilaterals. As examples of applications, we propose pseudospectral methods for two model problems and prove their spectral accuracy in space. Numerical results demonstrate the efficiency of the suggested algorithms. The approximation results and techniques developed in this paper are also applicable to other problems defined on quadrilaterals.

1. Introduction

During the past three decades, the spectral method has gained increasing popularity in scientific computations; see [1–11] and the references therein. The standard spectral method is traditionally confined to periodic problems and problems defined on rectangular domains. However, many practical problems are set on complex domains, for which we usually used the finite element method. For obtaining accurate numerical results, it is also interesting to consider spectral method and other high order methods for non-rectangular domains. Some authors proposed spectral methods for triangles and quadrilaterals; see, e.g., [1,2,5,12–15].

Recently, Guo and Jia [16,17] provided a spectral method for quadrilaterals and a spectral element method for polygons. But, it is more preferable to use pseudospectral method in actual computation, since we only need to evaluate unknown functions at interpolation nodes. This feature simplifies calculations and saves a lot of work, cf. [2,5,13].

The main difficulty of pseudospectral method for quadrilaterals is how to design proper numerical quadratures. In fact, for rectangle domains, it is natural to take the products of weights of one-dimensional numerical quadratures as the weights of two-dimensional numerical quadratures. In this case, the corresponding numerical quadratures keep the exactness for two-dimensional polynomials, which are taken as the basis functions of the pseudospectral method. Whereas, in pseudospectral method for quadrilaterals, the basis functions are induced by two-dimensional polynomials, with certain variable transformations changing various quadrilaterals to a standard square. Thus, they are not polynomials generally. Consequently, the exactness is no longer valid for such basis functions. Moreover, even if the exactness of numerical quadratures holds for some basis functions, it might fail for derivatives of basis functions. Furthermore, a reasonable pseudospectral method using certain numerical quadrature also depends on underlying problems. In other words, we should design different numerical quadratures for different differential equations, so that they possess the exactness for all terms involved in differential equations.

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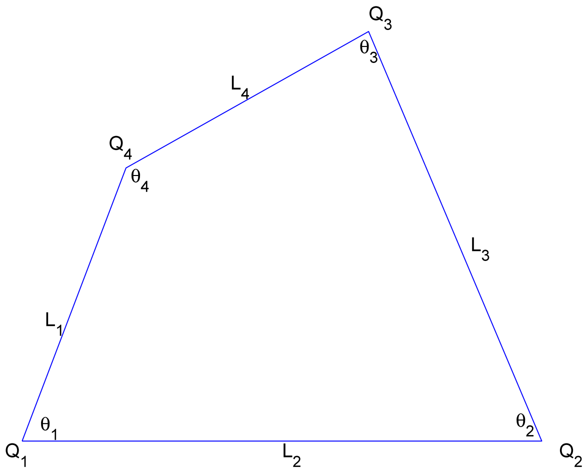

Fig. 1. Quadrilateral Ω.

This paper is devoted to the pseudospectral method on convex quadrilaterals. The next section is for preliminaries. In Section 3, we build up some basic results on the orthogonal approximation on quadrilaterals, which not only have the spectral accuracy, but also keep the high accuracy for a function possessing certain singularity at the edges or corners of quadrilateral. In Section 4, we establish the basic results on the Legendre–Gauss-type interpolation on quadrilaterals, which play important roles in designing and analyzing the pseudospectral method for various partial differential equations defined on quadrilaterals. As examples of applications, we provide pseudospectral schemes for two model problems in Section 5, and then prove their spectral accuracy in space. In Section 6, we present some numerical results indicating the high accuracy of the suggested algorithms. The final section is for concluding remarks.

2. Preliminaries

Let Ω be a convex quadrilateral with edges L_j, vertices $Q_j = (x_j, y_j)$, and angles θ_j, $1 \le j \le 4$, see Fig. 1. We make the variable transformation (cf. [2,5,16,13]):

$$x = a_0 + a_1\xi + a_2\eta + a_3\xi\eta, \qquad y = b_0 + b_1\xi + b_2\eta + b_3\xi\eta \tag{2.1}$$

where

$$\begin{aligned}
a_0 &= \frac{1}{4}(x_1 + x_2 + x_3 + x_4), \qquad b_0 = \frac{1}{4}(y_1 + y_2 + y_3 + y_4),\\
a_1 &= \frac{1}{4}(-x_1 + x_2 + x_3 - x_4), \qquad b_1 = \frac{1}{4}(-y_1 + y_2 + y_3 - y_4),\\
a_2 &= \frac{1}{4}(-x_1 - x_2 + x_3 + x_4), \qquad b_2 = \frac{1}{4}(-y_1 - y_2 + y_3 + y_4),\\
a_3 &= \frac{1}{4}(x_1 - x_2 + x_3 - x_4), \qquad b_3 = \frac{1}{4}(y_1 - y_2 + y_3 - y_4).
\end{aligned} \tag{2.2}$$

Then Ω is changed to the square $S = \{(\xi, \eta) \mid -1 < \xi, \eta < 1\}$. If Ω is a parallelogram, then $a_3 = b_3 = 0$. In this case, the transformation (2.1) is an affine mapping. Especially, for any rectangle Ω, we have $a_2 = a_3 = b_1 = b_3 = 0$. For simplicity, we denote $\frac{\partial x}{\partial \xi}$ by $\partial_\xi x$, etc. The Jacobi matrix of transformation (2.1) is

$$M_\Omega = \begin{pmatrix} \partial_\xi x & \partial_\xi y \\ \partial_\eta x & \partial_\eta y \end{pmatrix} = \begin{pmatrix} a_1 + a_3\eta & b_1 + b_3\eta \\ a_2 + a_3\xi & b_2 + b_3\xi \end{pmatrix}.$$

Its Jacobian determinant is

$$J_\Omega(\xi, \eta) = \begin{vmatrix} a_1 + a_3\eta & b_1 + b_3\eta \\ a_2 + a_3\xi & b_2 + b_3\xi \end{vmatrix} = d_0 + d_1\xi + d_2\eta \tag{2.3}$$

where

$$d_0 = a_1b_2 - a_2b_1, \qquad d_1 = a_1b_3 - a_3b_1, \qquad d_2 = a_3b_2 - a_2b_3.$$

According to (2.7) of [16,18], there exist positive constants δ_Ω and δ^*_Ω, such that

$$0 < \delta_\Omega \le J_\Omega(\xi, \eta) \le \delta^*_\Omega. \tag{2.4}$$