Journal of Computational and Applied Mathematics 235 (2011) 2006–2014

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A boundary-value problem for the polarized-radiation transfer equation with Fresnel interface conditions for a layered medium

A.E. Kovtanyuk, I.V. Prokhorov *

Institute of Applied Mathematics, Russian Academy of Sciences, Far-Eastern Division, Radio Street, 7, Vladivostok, 690041, Russia

ARTICLE INFO

Article history:
Received 9 June 2009

Keywords:
Vector transfer equation
Polarized radiation
Fresnel interface conditions
Monte Carlo method

ABSTRACT

A boundary-value problem for the polarized-radiation transfer equation for a layered medium with Fresnel matching conditions at the boundaries of the medium partition is examined. The theorems of solvability of the boundary-value problem are proved, and the continuity properties for its solution are examined. A numerical algorithm based on the Monte Carlo method for solving the boundary-value problem is proposed and proved.

In this work, a boundary problem for the polarized-radiation transfer equation for a layered medium is examined. Theoretical and numerical aspects of the vector radiation transfer equation solution were considered in [1–6]. In particular, in these works, general properties of the boundary problem solution were examined, and numerical and analytical methods for finding it were proposed.

The description of effects which appear at the boundaries between different materials is important for the simulation of radiation transfer within a substance. In radiation transfer theory these effects are taken into account through different matching conditions at the boundaries of the medium partition. The boundary problems for the scalar radiation transfer equation with continuous matching conditions for the solution at the boundaries of the medium partition are sufficiently researched [7–9].

Boundary problems with more general matching conditions for the solution, which describe reflection and refraction at the boundaries of the medium division, have been less examined. However, from the fifties in the last century specialists [10–12] repeatedly paid attention to these and different approaches for finding the boundary-value problem solution were proposed [1,4,11,13]. In the papers [14–17] the adequacy of the radiation transfer equation for realistic processes was examined, and its diffusion approximation was proved. In works [18,19], properties of the scalar radiation transfer equation solution with general matching conditions for 1D and 3D were examined.

In the present work, the main results of paper [19] are generalized to the vector case. The properties of continuity of the boundary problem solution for the polarized-radiation transfer equation for the layered medium are examined. Theorems of boundary problem solubility and estimates of the maximum principle type are obtained. A numerical algorithm based on the Monte Carlo method for solving the boundary-value problem is proposed. Numerical calculations, which demonstrate the influence of refraction, reflection and scattering on the polarization and depolarization of the radiation, are considered.

1. Problem formulation; main contingencies

Let the set G_0 be some partition of the set $G = (z_0, z_p)$ within which the radiation transfer process is examined:

$$G_0 = \bigcup_{i=1}^{p} G_i, \quad G_i = (z_{i-1}, z_i).$$

* Corresponding author.
E-mail addresses: ankov@imcs.dvgu.ru (A.E. Kovtanyuk), prh@iam.dvo.ru (I.V. Prokhorov).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2010.10.001

The planes $z = z_i$ are the interfaces between the layers G_i. Let us consider the equation of radiation transfer in a layered medium for the azimuthal symmetry case:

$$\nu f_z(z,\nu) + \mu(z) f(z,\nu) = \mu_s(z) \int_{-1}^{1} P(z,\nu,\nu') f(z,\nu') \mathrm{d}\nu' + J(z,\nu). \tag{1}$$

Here, $f(z,\nu) = (f_1(z,\nu), f_2(z,\nu))$ is a two-component vector of polarized radiation at the point $z \in G$ in the direction whose cosine of the angle with the positive direction of the axis z is $\nu \in [-1, 1]$. $f(z,\nu)$ is associated with the vector of the Stokes parameter $(I_\parallel, I_\perp)$ by the following relations:

$$f_1(z,\nu) = \frac{I_\parallel(z,\nu)}{n^2(z)}, \qquad f_2(z,\nu) = \frac{I_\perp(z,\nu)}{n^2(z)}.$$

Here, $n(z)$ is the piecewise-constant refractive index of the medium ($n(z) = n_i$ for $z \in G_i$). The sum $f_1 + f_2$ describes the density of the radiation flux, $f_1 \geq 0, f_2 \geq 0$. The functions μ, μ_s are called respectively the attenuation factor and the scattering coefficient. The two-component vector J describes internal radiation sources, and P is the 2×2 scattering matrix.

As for the coefficients in (1), we assume the following. Functions μ, μ_s, J_i are nonnegative; $\mu \geq \mu_{\min} > 0$, and $\mu, \mu_s \in C_b(G_0)$, where $C_b(G_0)$ is the Banach space of functions, bounded and continuous on G_0, with the norm

$$\|\varphi\|_{C_b(G_0)} = \sup_{x \in G_0} |\varphi(x)|.$$

We define $X = G \times \{[-1, 0) \cup (0, 1]\}$, $X_0 = G_0 \times \{[-1, 0) \cup (0, 1]\}$. We assume that all the components in the matrix P belong to $C_b(X_0 \times [-1, 1] \setminus \{0\})$, $(Pf)_{1,2} \geq 0$ for $f_{1,2} \geq 0$, and

$$\int_{-1}^{1} (p_{i1}(z,\nu,\nu') + p_{i2}(z,\nu,\nu')) \mathrm{d}\nu' = 1, \quad i = 1, 2.$$

We define the space $V(X_0)$ formed by the two-component vector functions $\varphi = (\varphi_1, \varphi_2)$, $\varphi_i \in C_b(X_0)$, with the norm

$$\|\varphi\|_{V(X_0)} = \max_{i=1,2} \|\varphi_i\|_{C_b(X_0)},$$

and let $J \in V(X_0)$.

We introduce the following boundary sets:

$$\Gamma_{\text{int}} = \bigcup_{i=1}^{p-1} \{z_i \times \{[-1, 0) \cup (0, 1]\}\}, \qquad \Gamma_{\text{ext}}^{\pm} = \{\{z_0 \times [\mp 1, 0)\} \cup \{z_p \times [\pm 1, 0)\}\},$$
$$\Gamma^{\pm} = \Gamma_{\text{int}} \cup \Gamma_{\text{ext}}^{\pm}, \quad \Gamma = \Gamma^+ \cup \Gamma^-.$$

We supplement Eq. (1) with the boundary condition

$$f^-(z,\nu) = (Bf^+)(z,\nu) + h(z,\nu), \quad (z,\nu) \in \Gamma^-, \tag{2}$$

where

$$f^{\pm}(z,\nu) = \begin{cases} f(z \pm 0, \nu), & \nu < 0, \\ f(z \mp 0, \nu), & \nu > 0, \end{cases}$$
$$f(z \pm 0, \nu) = \lim_{\varepsilon \to +0} f(z \pm \varepsilon, \nu).$$

The function h describes the radiation flux that enters the medium G. Let $h \in V(\Gamma^-)$, and $h = 0$ at the set Γ_{int}. The operator B defines matching conditions for the set Γ_{int}, and for the set Γ_{ext} we assume $B = 0$. Thus, equality (2) defines boundary conditions both at the external part of the set G_0 and at its internal boundaries.

We introduce the functions

$$\widetilde{n}_i(\nu) = \begin{cases} n_i/n_{i-1}, & \text{for } 0 < \nu \leq 1, \\ n_{i-1}/n_i, & \text{for } -1 \leq \nu < 0, \end{cases}$$

$$\psi_i(\nu) = \begin{cases} \text{sgn}(\nu)\sqrt{1 - \widetilde{n}_i^2(\nu)(1-\nu^2)}, & \text{for } 1 - \widetilde{n}_i^2(\nu)(1-\nu^2) \geq 0, \\ 0, & \text{for } 1 - \widetilde{n}_i^2(\nu)(1-\nu^2) < 0. \end{cases}$$

We define a matching operator B that will be used to model Fresnel reflection and refraction at the contact boundaries z_i, $i = \overline{1, p-1}$. Let

$$(Bf^+)(z_i,\nu) = R_i(\nu) f^+(z_i, \nu_R) + T_i(\nu) f^+(z_i, \nu_T),$$