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Solvability of Newton equations in smoothing-type algorithms for the SOCCP☆

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ABSTRACT

In this paper, we first investigate the invertibility of a class of matrices. Based on the obtained results, we then discuss the solvability of Newton equations appearing in the smoothing-type algorithm for solving the second-order cone complementarity problem (SOCCP). A condition ensuring the solvability of such a system of Newton equations is given. In addition, our results also show that the assumption that the Jacobian matrix of the function involved in the SOCCP is a P_0-matrix is not enough for ensuring the solvability of such a system of Newton equations, which is different from the one of smoothing-type algorithms for solving many traditional optimization problems in $\Re^n$.

1. Introduction

It is well known that many optimization problems can be reformulated as a system of parameterized smooth equations. Instead of solving the original problem, one solves the parameterized equations by some Newton-type method that iteratively finds a solution of the smooth equations while gradually reducing the smoothing parameter to zero so that a solution of the original problem can be found. This is the so-called smoothing-type algorithm, which has been successfully applied to various optimization problems (see, for example, [1–12]). In order to ensure the well-definedness of some smoothing-type algorithm, it is fundamental to ensure the solvability of Newton equations appearing in the smoothing-type algorithm.

In the smoothing-type algorithm for many optimization problems in $\Re^n$, the solvability of Newton equations is usually determined by the invertibility of the matrix of the form

$$\bar{N} = \begin{pmatrix} M & -I \\ X & Y \end{pmatrix},$$

where $M, I, X, Y \in \Re^{n\times n}$, I is the identity matrix, and both X and Y are positive diagonal matrices. Kojima et al. showed in [13, Lemma 4.1] that $\bar{N}$ is invertible if and only if M is a P_0-matrix (i.e., for every $0 \neq x \in \Re^n$, there exists an $x_k \neq 0$ such that $x_k(Mx)_k \geq 0$). Such a result plays an important role in some algorithms for solving many optimization problems in $\Re^n$, such as smoothing-type algorithms for solving complementarity problems (CPs) [1–3,5] and variational inequality problems (VIPs) [6–9].

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The second-order cone complementarity problem (SCCCP) is to find an $x = (x_1, x_2) \in \Re \times \Re^{n-1}$ such that

$$x \succeq 0, \quad f(x) + q \succeq 0, \qquad \langle x, f(x) + q \rangle = 0, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, $f : \Re^n \to \Re^n$, and $\succeq$ is a partial order induced by

$$\mathcal{K} := \mathcal{K}^{n_1} \times \mathcal{K}^{n_2} \times \cdots \times \mathcal{K}^{n_m} \tag{1.2}$$

(i.e., $x \succeq 0$ means $x \in \mathcal{K}$. Similarly, $x \succ 0$ means $x \in \text{int } \mathcal{K}$, the interior of $\mathcal{K}$), where integers $m \geq 0, n_1, \ldots, n_m \geq 0$, $n_1 + \cdots + n_m = n$, and every $\mathcal{K}^{n_i}$ is a second-order cone defined by $\mathcal{K}^{n_i} := \{(x_1, x_2) \in \Re \times \Re^{n_i - 1} : \|x_2\| \leq x_1\}$ with $\| \cdot \|$ denoting the Euclidean norm. The SOCCP has been studied extensively in the literature (see, for example, [14–19]). In this paper, unless stated otherwise, we assume that $\mathcal{K} = \mathcal{K}^n$. We shall show that, in smoothing-type algorithms for the SOCCP, the solvability of Newton equations is determined by the invertibility of the matrix in the form of

$$N := \begin{pmatrix} M & -I \\ X & Y \end{pmatrix}, \tag{1.3}$$

where I is the identity matrix, M is the Fréchet derivative of f at x, and $(X, Y) \in \Omega_1$ with Ω_1 being defined by

$$\Omega_1 := \left\{ (X, Y) \in \Re^{n\times n} \times \Re^{n \times n} \middle| \begin{array}{l} X, Y \text{ are two symmetric positive} \\ \text{definite matrices and } XY = YX \end{array} \right\}. \tag{1.4}$$

Thus, it is necessary to investigate the invertibility of the matrix N defined by (1.3) in order to develop smoothing-type algorithms to solve the SOCCP. *A natural question is whether the result on the invertibility of the matrix $\tilde{N}$ can be extended to the matrix N or not? If not, which condition for M can ensure that the matrix N defined by (1.3) with $(X, Y) \in \Omega_1$ is invertible?*

In this paper, we show that N is invertible for any $(X, Y) \in \Omega_1$ if and only if $M \in \Omega_2$ where Ω_2 is defined by

$$\Omega_2 := \{M \in \Re^{n\times n} : QMQ^T \text{ is a } P_0\text{-matrix for any orthogonal matrix } Q\}, \tag{1.5}$$

and that $M \in \Omega_2$ if and only if M is a positive semidefinite matrix. As mentioned above, we shall show that the solvability of Newton equations is determined by the invertibility of the matrix in the form of N defined by (1.3). In particular, such a system of Newton equations is solvable if M is positive semidefinite. Our results also show that the assumption that M is a P_0-matrix is not enough to ensure the solvability of such a system of Newton equations, which is different from the one of the existing smoothing-type algorithms for solving many optimization problems in $\Re^n$.

The rest of this paper is organized as follows. In Section 2, we show that a matrix belongs to Ω_2 defined by (1.5) if and only if such a matrix is a positive semidefinite matrix, and discuss the invertibility of the matrix N with $(X, Y) \in \Omega_1$. In Section 3, we discuss the solvability of Newton equations appearing in the smoothing-type algorithm for the SOCCP. Some remarks are also given in this section.

In our notation, all vectors are column vectors, $\mathcal{I} := \{1, 2, \ldots, n\}$, the superscript T denotes transpose, $\Re^n$ denotes the space of n-dimensional real column vectors, $\Re^{n\times n}$ denotes the space of $n \times n$ real matrices, and $Df(x)$ denotes the Fréchet derivative of $f(\cdot) : \Re^n \to \Re^n$ at x. For any vectors $u, v \in \Re^n$, we denote by u_i the ith component of u, and write $(u^T, v^T)^T$ as (u, v) for simplicity.

2. Invertibility of the matrix *N*

In this section, we show that a matrix belongs to Ω_2 defined by (1.5) if and only if such a matrix is a positive semidefinite matrix, and then discuss the invertibility of the matrix N defined by (1.3) with $(X, Y) \in \Omega_1$ defined by (1.4).

We first recall some basic concepts and results.

Definition 2.1. Given $M \in \Re^{n\times n}$ and $f : \Re^n \to \Re^n$.

(i) M is called a positive semidefinite matrix if $x^T Mx \geq 0$ for every $x \in \Re^n$; and a P_0-matrix if for every $0 \neq x \in \Re^n$, there exists $x_k \neq 0$ such that $x_k(Mx)_k \geq 0$.
(ii) f is a monotone function if for any $x, y \in \Re^n$, $\langle x - y, f(x) - f(y)\rangle \geq 0$; and a P_0-function if for every $x \neq y \in \Re^n$, there exists $x_k \neq y_k$ such that $(x_k - y_k)(f(x) - f(y))_k \geq 0$.

Proposition 2.1. *Given $M \in \Re^{n\times n}$ and $f : \Re^n \to \Re^n$, the following are known.*

(i) *$M \in \Re^{n\times n}$ is a P_0-matrix if and only if all its principal minors are nonnegative.*
(ii) *If $f : \Re^n \to \Re^n$ is Fréchet differentiable, then f is a monotone function if and only if $Df(x)$ is a positive semidefinite matrix for any $x \in \Re^n$.*

By using Definition 2.1 and Proposition 2.1, we establish the following necessary and sufficient condition.

Theorem 2.1. *$M \in \Omega_2$ (see (1.5)) if and only if M is a positive semidefinite matrix.*