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Uzawa block relaxation method for the unilateral contact problem

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A B S T R A C T

We present a Uzawa block relaxation method for the numerical resolution of contact problems with or without friction, between elastic solids in small deformations. We introduce auxiliary unknowns to separate the linear elasticity subproblem from the unilateral contact and friction conditions. Applying a Uzawa block relaxation method to the corresponding augmented Lagrangian functional yields a two-step iterative method with a linear elasticity problem as a main subproblem while auxiliary unknowns are computed explicitly. Numerical experiments show that the method are robust and scalable with a significant saving of computational time.

1. Introduction

The contact problem is a very common problem in engineering (rails, gear, forming, etc.). The presence of unilateral and friction constraints poses a serious challenge compared with the classical linear elasticity problem. Various numerical methods for solving unilateral contact problems with or without friction have been developed during the past decades. For frictionless unilateral contact problems, we refer, e.g., to [1–3] and the references therein. For the unilateral contact problem with the Coulomb friction, two main approaches can be considered:

- the direct approach, i.e. the system of discretized equations is solved; see e.g. [4–11].
- The Coulomb friction as the limit of a sequence of the Tresca friction problems; see e.g. [12–14,7,1,15].

Direct approaches are based on finite-dimensional problems and their implementation can be complicated. The second approach is more commonly used and requires fast methods for solving the Tresca friction problems; see e.g. [13–15,2].

The method proposed in this paper is related to the augmented Lagrangian operator-splitting methods; see e.g. [16,17]. The main idea is to separate the linear part of the problem (i.e. linear elasticity) from the nonlinear part (i.e. unilateral contact and friction conditions) by introducing auxiliary variables. Applying a Uzawa block relaxation type method to the corresponding augmented Lagrangian leads to a simple two-step iterative method. In the first step a linear elasticity problem is solved. In the second step, the auxiliary variables are computed explicitly using the duality theory. The main advantage of our method is that the matrix of the linear elasticity problem solved in the first step is constant during the iterative process, saving computational time due to matrix factorizations.

The paper is organized as follows. In Section 2 the model problem is presented followed by its augmented Lagrangian formulation in Section 3. The Uzawa block relaxation algorithms for frictionless and friction cases are presented in Sections 4 and 5, respectively. The convergence theorem of the algorithm is presented in Section 6. Numerical experiments on two model examples are presented in Section 7.

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2. The model problem

We consider an elastic body occupying in its initial (undeformed) configuration a bounded domain Ω of $\mathbb{R}^2$ with a boundary $\Gamma = \Gamma_D \cup \Gamma_c$. We assume that the elastic body is fixed along Γ_D with meas$(\Gamma_D) > 0$. Γ_c denotes a portion of Γ which is a candidate contact surface between Ω and a rigid foundation. The normalized gap between Γ_c and the rigid foundation is denoted by g. In this paper, we consider the small strain hypothesis so that the strain tensor is $\epsilon(u) = (\nabla u + \nabla u^t)/2$, where $u = (u_1(x), u_2(x))$ is the displacement field. Hooke's law is assumed, i.e. the stress tensor is linked to the displacement through the linear relation

$$\sigma(u) = C\epsilon(u)$$

where $C = (C_{ijkl})$ is the (fourth order) elastic moduli tensor, assumed to be symmetric positive definite. Let n be the outward unit normal to Ω on Γ. It is usual to decompose the displacement field and the stress tensor in normal and tangential components:

$$u_n = u \cdot n, \qquad u_t = u - u_n n, \tag{2.1}$$

$$\sigma_n(u) = (\sigma(u)n) \cdot n, \qquad \sigma_t(u) = \sigma(u) - \sigma_n n. \tag{2.2}$$

The unilateral contact problem with Coulomb friction consists, for a given volume force f, of finding the displacement field u satisfying

(i) the equilibrium equations

$$-\operatorname{div} \sigma(u) = f \quad \text{in } \Omega, \tag{2.3}$$

$$u = 0 \quad \text{on } \Gamma_D, \tag{2.4}$$

(ii) the contact (i.e. non-penetration) conditions

$$u_n - g \leq 0, \qquad \sigma_n(u) \leq 0, \qquad (u_n - g)\sigma_n(u) = 0, \quad \text{on } \Gamma_c, \tag{2.5}$$

(iii) the Coulomb friction conditions

$$|\sigma_t(u)| \leq \nu_f |\sigma_n(u)|, \qquad |\sigma_t(u)| < \nu_f |\sigma_n(u)| \Longrightarrow u_t = 0 \quad \text{on } \Gamma_c, \tag{2.6}$$

$$|\sigma_t(u)| = \nu_f |\sigma_n(u)| \Longrightarrow \exists \lambda \geq 0, \qquad u_t = -\lambda \sigma_t(u) \quad \text{on } \Gamma_c. \tag{2.7}$$

In (2.6)–(2.7), ν_f stands for the (positive) friction coefficient.

If the normal stress $\sigma_n(u)$ on Γ_c is known, the Coulomb friction conditions (2.6)–(2.7) can be replaced by the Tresca friction conditions

$$s = \nu_f |\sigma_n(u)|, \qquad |\sigma_t(u)| < s \implies u_t = 0 \quad \text{on } \Gamma_c, \tag{2.8}$$

$$|\sigma_t(u)| = s \implies \exists \lambda \geq 0, \qquad u_t = -\lambda \sigma_t(u) \quad \text{on } \Gamma_c. \tag{2.9}$$

In this paper, we approximate the Coulomb friction (2.6)–(2.7) by solving a sequence of Tresca friction problems. Uzawa block relaxation algorithms are therefore designed for the unilateral frictionless contact problems (2.3)–(2.5) and the unilateral contact problem with Tresca friction (2.3)–(2.5), (2.8)–(2.9).

3. Augmented Lagrangian formulation

The contact problem with Tresca friction can be stated as an optimization problem allowing the use of arguments from convex analysis and duality theory to show the existence of a unique solution and to design numerical algorithms. Let us introduce space of functions

$$V = \left\{ v \in H^1(\Omega)^2,\ v = 0 \text{ on } \Gamma_D \right\}$$

and the set of admissible displacements

$$K = \{ v \in V,\ v_n - g \leq 0 \text{ on } \Gamma_c \}.$$

Let $a(\cdot, \cdot)$ be the symmetric, continuous and coercive bilinear form which corresponds to the virtual work in the elastic body

$$a(u, v) = \int_\Omega \sigma_{ij}(u)\epsilon_{ij}(v)\mathrm{d}x.$$

We denote by $f(\cdot)$ the linear form of external forces

$$f(v) = \int_\Omega f \cdot v \mathrm{d}x.$$