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Improved rectangular method on stochastic Volterra equations

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ABSTRACT

An improved version of rectangular method (IRM) is introduced in this paper to numerically solve the stochastic Volterra equation (SVE). We focus on studying the order of error between the numerical and exact solutions, which is improved to $O(h)$. Furthermore, an explicit form of the IRM scheme is introduced and its convergence is established. A numerical example has also been presented to show the feasibility of the methods.

1. Introduction

Let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space with a filtration $\{\mathcal{F}_t, t \geq 0\}$ that is increasing and right continuous and $\mathcal{F}_0$ contains all P-null sets. Let W_t be a standard Brownian motion defined on the probability space. The aim of this paper is to introduce numerical schemes to Itô type stochastic Volterra equation (SVE) of the form:

$$X_t = x + \int_0^t a(t, X_s)\mathrm{d}s + \int_0^t b(t, X_s)\mathrm{d}W_s, \quad t \in [0, T], \tag{1.1}$$

where a, b are measurable functions, x is the initial value. Stochastic Volterra equations are a natural extension of deterministic ones. Comprehensive results concerning deterministic Volterra equations and their numerical solutions in connection with problems arising in mathematical physics can be found in the existing literature (see [1–5] for example).

Recently, stochastic Volterra equations have received great attention (see [6–9] for example). Most SVEs do not have analytic solutions and hence it is of great importance to provide numerical schemes. Numerical schemes to stochastic differential equations (SDEs) have been well developed (see [10–13] for example). However, there are still very few papers discussing the numerical solutions for stochastic Volterra equations.

A linear stochastic difference equation was introduced in [14] to solve a linear stochastic Volterra integro-differential equation. Their method employed the Euler scheme to approximate the stochastic differential part and a Θ method to approximate the integral with a quadrature. Their numerical method works only for a special case of stochastic Volterra equations. Numerical solutions to Volterra equations in Hilbert spaces have also been studied recently. Galerkin method was

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used in [15] to address the numerical problem of a particular linear Volterra equation. In this paper, the authors adapted a resolvent approach to treat a special class of stochastic Volterra equations.

In [16], we have proposed a rectangular method (RM) for the stochastic Volterra equation of the form (1.1), and establish the order $O(\sqrt{h})$ between the numerical and exact solutions. In this paper, we will develop an improved rectangular method (IRM) for (1.1) and this scheme allows to improve the order of error to $O(h)$. Furthermore, we will introduce an explicit form of IRM scheme, which avoids derivative in the same way as the Runge–Kutta schemes for deterministic cases and hence makes the scheme more feasible to implement. The convergence of the explicit IRM scheme is also established. For simplicity of exposition, only one-dimensional stochastic Volterra equations are considered in this paper. We like to stress that our methods work equally well also for multi-dimensional stochastic Volterra equations.

The rest of the paper is organised in three sections. Section 2 reviews the RM scheme and introduces the IRM scheme. Section 3 establishes the convergence of IRM scheme. Section 4 discusses the explicit form of IRM and its convergence theorem. We conclude the paper with numerical example in Section 5.

2. IRM scheme

Consider the one-dimensional SVE:

$$X_t = X_0 + \int_0^t a(t, X_s)\mathrm{d}s + \int_0^t b(t, X_s)\mathrm{d}W_s, \quad 0 \le t \le T, \tag{2.1}$$

with $X_0 = x$, where $a : R \times R \to R$ and $b : R \times R \to R$.

In this paper, we discuss the numerical schemes with uniform stepsize, that is $h = T/N, N \ge 1$. Let $t_n = nh, n = 0, 1, 2 \dots N$, with $[s]$ denoting the largest integer smaller than s. Let $n_s = [s/h]$, and $s_n = [s/h]h = t_{n_s}$. Furthermore, for convenience we write X_{t_n} as X_n and its approximation as Y_n. We use the following notations: $a'_t(s, X_u) = \frac{\partial a}{\partial s}(s, X_u)$, $a''_t(s, X_u) = \frac{\partial a^2}{\partial s^2}(s, X_u)$, $a'_x(s, X_u) = \frac{\partial a}{\partial X_u}(s, X_u)$, $a''_{tx}(s, X_u) = \frac{\partial a^2}{\partial s \partial X_u}(s, X_u)$ and etc. C and K are constants throughout this paper but they may change from line to line.

It is well known [16] that under the following hypotheses:

(H1): X_0 is $\mathcal{F}_0$-measurable with $E[|X_0|^2] < \infty$

(H2): (Global Lipschitz condition) There exists a constant $K > 0$ such that

$$\begin{aligned}
&|a(t, x) - a(t, y)| \vee |b(t, x) - b(t, y)| \le K|x - y|;\\
&|a(t, x) - a(s, x)|^2 \vee |b(t, x) - b(s, x)|^2 \le K(1 + |x|^2)|t - s|,\\
&\text{for all } s \le t \in [0, T] \quad \text{and} \quad x \in R,
\end{aligned}$$

the stochastic Volterra equation (2.1) has a continuous pathwise-unique strong solution X_t on $[0, T]$ such that

$$\begin{aligned}
&\sup_{0 \le t \le T} E[|X_t|^2] \le CE(1 + |X_0|^2);\\
&E(|X_t - X_s|^2) \le CE(1 + |X_0|^2)|t - s|.
\end{aligned}$$

Without loss of generality, we only discuss the approximate solution at grid points t_i, where $i \ge 0$. In [16], we have proposed rectangular method (RM) to SVEs which is defined as follows:

Let $Y_0 = X_0$, and

$$Y_n = Y_0 + \sum_{i=0}^{n-1} a(t_n, Y_i)h + \sum_{i=0}^{n-1} b(t_n, Y_i)\Delta W_i, \tag{2.2}$$

where $h = t_{i+1} - t_i$ and $\Delta W_i = W_{i+1} - W_i$.

Inspired by the modification of Euler scheme to Milstein scheme, we will introduce IRM by adding two more terms in each approximate step to decrease the local error for approximate solution. However unlike Milstein scheme, the drift and volatility coefficients of approximate solution will be updated all the time due to the nature of long time dependence in the Volterra equation.

We propose the improved rectangular method (IRM) for the Volterra equation as follows:

Let $Y_0 = X_0$.

$$\begin{aligned}
Y_1 = {}& Y_0 + \int_0^{t_1} a(t_1, Y_0)\mathrm{d}u + \int_0^{t_1} b(t_1, Y_0)\mathrm{d}W_s\\
&+ \int_0^{t_1}\int_0^s a'_x(t_1, Y_0)b(0, Y_0)\mathrm{d}W_u\mathrm{d}s + \int_0^{t_1}\int_0^s b'_x(t_1, Y_0)b(0, Y_0)\mathrm{d}W_u\mathrm{d}W_s.
\end{aligned} \tag{2.3}$$