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On CSCS-based iteration methods for Toeplitz system of weakly nonlinear equations

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ABSTRACT

For Toeplitz system of weakly nonlinear equations, by using the separability and strong dominance between the linear and the nonlinear terms and using the circulant and skew-circulant splitting (CSCS) iteration technique, we establish two nonlinear composite iteration schemes, called Picard-CSCS and nonlinear CSCS-like iteration methods, respectively. The advantage of these methods is that they do not require accurate computation and storage of Jacobian matrix, and only need to solve linear sub-systems of constant coefficient matrices. Therefore, computational workloads and computer storage may be saved in actual implementations. Theoretical analysis shows that these new iteration methods are local convergent under suitable conditions. Numerical results show that both Picard-CSCS and nonlinear CSCS-like iteration methods are feasible and effective for some cases.

1. Introduction

Consider iterative solution of the following large Toeplitz system of weakly nonlinear equations

$$Ax = \phi(x), \quad \text{or} \quad F(x) := Ax - \phi(x) = 0, \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}$ is a large, nonsymmetric and positive definite Toeplitz matrix, $\phi : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is a continuously differentiable function defined on the open convex domain $\mathbb{D}$ in the n-dimensional linear space $\mathbb{C}^n$. Here, the system of nonlinear equations (1.1) is said to be Toeplitz weakly nonlinear if the linear term Ax is strongly dominant over the nonlinear term $\phi(x)$ in certain norm and A is a Toeplitz matrix; see [1–3].

The system of weakly nonlinear equations (1.1) may arise in many areas of scientific computing and engineering applications. For example, in finite-difference or sinc discretizations of nonlinear partial differential equations [4–7], in collocation approximations of nonlinear integral equation [8] and in saddle point problems from image processing [9,10].

A matrix A is said to be Toeplitz if

$$A = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{2-n} & a_{1-n} \\ a_1 & a_0 & a_{-1} & \cdots & a_{2-n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2} & \cdots & a_1 & a_0 & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix},$$

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i.e., A is constant along its diagonals; see [11]. A Toeplitz matrix A possesses a circulant and skew-circulant splitting $A = C+S$, where

$$C = \frac{1}{2}\begin{bmatrix} a_0 & a_{-1}+a_{n-1} & \cdots & a_{2-n}+a_2 & a_{1-n}+a_1 \\ a_1+a_{1-n} & a_0 & \cdots & \cdots & a_{2-n}+a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2}+a_{-2} & \cdots & \cdots & a_0 & a_{-1}+a_{n-1} \\ a_{n-1}+a_{-1} & a_{n-2}+a_{-2} & \cdots & a_1+a_{1-n} & a_0 \end{bmatrix} \quad (1.2)$$

and

$$S = \frac{1}{2}\begin{bmatrix} a_0 & a_{-1}-a_{n-1} & \cdots & a_{2-n}-a_2 & a_{1-n}-a_1 \\ a_1-a_{1-n} & a_0 & \cdots & \cdots & a_{2-n}-a_2 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-2}-a_{-2} & \cdots & \cdots & a_0 & a_{-1}-a_{n-1} \\ a_{n-1}-a_{-1} & a_{n-2}-a_{-2} & \cdots & a_1-a_{1-n} & a_0 \end{bmatrix}. \quad (1.3)$$

Note that C is a circulant matrix and S is a skew-circulant matrix. A circulant matrix can be diagonalized by the discrete Fourier matrix F and a skew-circulant matrix can be diagonalized by a discrete Fourier matrix with diagonal scaling, i.e., $\hat{F} = F\Omega$. That is to say, it holds that

$$F^*CF = \Lambda_C, \qquad \hat{F}^*S\hat{F} = \Lambda_S, \quad (1.4)$$

where

$$F = (F)_{j,k} = \frac{1}{\sqrt{n}}e^{\frac{2\pi \mathbf{i}}{n}jk}, \quad 0 \le j,k \le s, \qquad \Omega = \mathrm{diag}\left(1, e^{-\frac{\pi\mathbf{i}}{n}}, \ldots, e^{-\frac{(n-1)\pi\mathbf{i}}{n}}\right),$$

and $\mathbf{i}$ is the imaginary unit [12,13]. Λ_C and Λ_S are diagonal matrices formed by the eigenvalues of C and S, respectively, which can be obtained in $O(n\log n)$ operations by using the fast Fourier transform (FFT).

As is known, the Newton method may be the most popular, classic and important solver for a general system of nonlinear equations $F(x) = 0$, where $F : \mathbb{D} \subset \mathbb{C}^n \to \mathbb{C}^n$ is a continuously differentiable function. However, at each iteration step, the Newton method requires not only the computation of $F(x^{(k)})$ and $F'(x^{(k)})$, but also the exact solution of the corresponding Newton equation $F'(x^{(k)})\Delta x^{(k)} = -F(x^{(k)})$, which are very costly and complicated in actual applications [14,15]. In order to overcome these disadvantages and improve the efficiency of the Newton iteration method, a large number of modifications have been proposed to simplify or avoid computation of the Jacobian matrix and reduce the cost of the function evaluation; see [3,14,16–19].

For the weakly nonlinear system (1.1), based on the facts that the linear and the nonlinear terms Ax and $\phi(x)$ are well separated and the former is strongly dominant over the latter, Bai and Yang [3] presented the Picard-HSS and the nonlinear HSS-like iteration methods. The advantage of these methods over the Newton iteration method is that they do not require explicit construction and accurate computation of the Jacobian matrix, and only need to solve linear sub-systems of constant coefficient matrices. Hence, computational workloads and computer memory may be saved.

In this paper, based on the circulant and skew-circulant splitting (CSCS) of the Toeplitz matrix, we establish two classes of nonlinear composite splitting iteration schemes, called Picard-CSCS and nonlinear CSCS-like iteration methods, respectively, for solving the large scale Toeplitz system of weakly nonlinear equations (1.1). Compared with the Newton iteration method, both Picard-CSCS and nonlinear CSCS-like iteration methods neither require explicit form and accurate computation of Jacobian matrix, nor require solution of the changeable-coefficient linear sub-systems, which is similar to the Picard-HSS and the nonlinear HSS-like iteration methods initially introduced in [3]. Moreover, as the circulant and skew-circulant matrices can be diagonalized by the discrete Fourier matrix and diagonally scaled discrete Fourier matrix, respectively, the solutions of the two linear sub-systems can be efficiently obtained by using FFT. In addition, FFT is highly parallelizable and has been implemented on multiprocessors efficiently [20]. Hence, computational workloads may be further saved in actual implementations.

The organization of this paper is as follows. In Section 2, we review the CSCS iteration and give the Newton-CSCS iteration method. In Sections 3 and 4, we establish the Picard-CSCS and the nonlinear CSCS-like iteration methods, and discuss their convergence properties. Numerical results are given in Section 5. Finally, in Section 6 we draw a brief conclusion and give some remarks.

2. The CSCS and Newton-CSCS iteration methods

When the nonlinear term $\phi : \mathbb{D} \in \mathbb{C}^n \to \mathbb{C}^n$ is a constant vector, i.e., $\phi(x) = b$, the Toeplitz system of weakly nonlinear equations (1.1) reduces to the Toeplitz system of linear equations

$$Ax = b, \quad A \in \mathbb{C}^{n\times n} \text{ and } x, b \in \mathbb{C}^n. \quad (2.1)$$