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Analytical and numerical treatment of oscillatory mixed differential equations with differentiable delays and advances

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ABSTRACT

In this work, we study the oscillatory behaviour of the differential equation of mixed type

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))\,d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))\,d\eta(\theta)$$

with delays $r(\theta)$ and advances $\tau(\theta)$, both differentiable. Some analytical and numerical criteria are obtained in order to guarantee that all solutions are oscillatory.

1. Introduction

The aim of this work is to study the oscillatory behaviour of the differential equation of mixed type

$$x'(t) = \int_{-1}^{0} x(t - r(\theta))d\nu(\theta) + \int_{-1}^{0} x(t + \tau(\theta))d\eta(\theta) \tag{1}$$

where $x(t) \in \mathbb{R}$, $\nu(\theta)$ and $\eta(\theta)$ are real functions of bounded variation on $[-1, 0]$ normalized so that $\nu(-1) = \eta(-1) = 0$, and $r(\theta)$ and $\tau(\theta)$ are nonnegative real continuous functions on $[-1, 0]$. Taking

$$\|\tau\| = \max\{\tau(\theta) : \theta \in [-1, 0]\},$$

the advance $\tau(\theta)$ will be assumed to satisfy

$$\tau(\theta_0) = \|\tau\| > \tau(\theta), \quad \forall \theta \neq \theta_0. \tag{2}$$

In the case of $\tau(\theta_0) > 0$, the function $\eta(\theta)$ is supposed to be atomic at θ_0, that is, such that

$$\eta(\theta_0^+) - \eta(\theta_0^-) \neq 0. \tag{3}$$

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The Eq. (1) represents the wider class of linear functional differential equations of mixed type and is considered in [1] as a basis for some mathematical applications appearing in the literature, such as in [2,3].

Letting $R = \max\{\|r\|, \|\tau\|\}$, by a solution of (1) we will mean any differentiable function $x : [-R, +\infty) \to \mathbb{R}$ which satisfies (1) for every $t \in [0, +\infty)$.

As usual, we will say that a solution $x(t)$ of (1) oscillates if it has arbitrarily large zeros. In [1] $x(t)$ is called oscillatory if there is no cone, $\mathcal{K}$, such that $x(t) \in \mathcal{K}$, eventually. Notice that for equations, both definitions coincide. When all solutions oscillate (1) will be said to be oscillatory.

By assuming that delays and advances are positive and differentiable on $[-1, 0]$, one can obtain some special criteria for having (1) oscillatory. In this paper we will analyse this case, complementing the results in [4] for the case where delays and advances are only continuous. Further theoretical results for delay equations are obtained in [5] and these can be extended in a natural way to the mixed equation.

The two main ingredients in theory of linear delay equations (see [6]) are the existence of a unique solution, for any given initial condition, and the exponential boundedness on those solutions. As is shown in [7], this is not at all the situation of a differential equation of mixed type like (1). However, under the atomicity assumption (3), one has that every oscillatory solution is exponentially bounded as $t \to \infty$ [1, Proposition 4]. This fact enables the oscillatory behaviour of (1) to be studied through the analysis of the zeros of the characteristic equation

$$\lambda = \int_{-1}^{0} \exp(-\lambda r(\theta))\mathrm{d}\nu(\theta) + \int_{-1}^{0} \exp(\lambda\tau(\theta))\mathrm{d}\eta(\theta). \tag{4}$$

In fact, if we let

$$M(\lambda) = \int_{-1}^{0} \exp(-\lambda r(\theta))\mathrm{d}\nu(\theta) + \int_{-1}^{0} \exp(\lambda\tau(\theta))\mathrm{d}\eta(\theta),$$

by [1, Corollary 5] the Eq. (1) is oscillatory if and only if $M(\lambda) \neq \lambda$, for every real λ. Therefore, if either

$$M(\lambda) > \lambda, \quad \forall \lambda \in \mathbb{R} \tag{5}$$

or

$$M(\lambda) < \lambda, \quad \forall \lambda \in \mathbb{R} \tag{6}$$

we can conclude that Eq. (1) is oscillatory.

2. Differentiable delays and advances

By an increasing (decreasing) function on an interval $[a, b]$ we will mean any nondecreasing (respectively nonincreasing) function, ϕ, such that $\phi(a) < \phi(b)$ (respectively, $\phi(a) > \phi(b)$). Assuming that $-1 \leqslant \theta_1 \leq 0$, let $D^+(\theta_1)$ be the family of all positive differentiable functions, which are increasing on $[-1, \theta_1]$ and decreasing on $[\theta_1, 0]$. If $\theta_1 = 0$, we obtain the set, D_i^+ of all positive increasing differentiable functions on the interval $[-1, 0]$. In the case where $\theta_1 = -1$, we obtain the class D_d^+ of all decreasing positive differentiable functions on $[-1, 0]$.

For $r \in D^+(\theta_1)$ and $\tau \in D^+(\theta_0)$ with θ_0 as in (2), we define the value

$$S_1 = \mathrm{e}^{-1}\left(\int_{-1}^{0} \nu(\theta)\mathrm{d}\ln r(\theta) + \int_{-1}^{0} \eta(\theta)\mathrm{d}\ln\tau(\theta)\right).$$

Through (5) we obtain the following theorems.

Theorem 2.1. *For $r \in D^+(\theta_1)$ and $\tau \in D^+(\theta_0)$, let*

$$\nu(\theta) \leqslant 0 \quad \text{for } \theta \in [-1, \theta_1[, \qquad \nu(\theta) \geqslant 0 \quad \text{for } \theta \in [\theta_1, 0] \tag{7}$$

$$\eta(\theta) \leqslant 0 \quad \text{for } \theta \in [-1, \theta_0[, \qquad \eta(\theta) \geqslant 0 \quad \text{for } \theta \in [\theta_0, 0], \tag{8}$$

such that $\eta(0) > 0$. If

$$1 + \ln(\tau(0)\eta(0)) + \tau(0)S_1 > 0 \tag{9}$$

then the Eq. (1) is oscillatory.

Proof. For $\lambda = 0$, we have $M(0) = \nu(0) + \eta(0) > 0$. Let $\lambda \neq 0$. Using integration by parts we obtain

$$M(\lambda) = \exp(-\lambda r(0))\nu(0) + \exp(\lambda\tau(0))\eta(0) + \lambda\int_{-1}^{0} \exp(-\lambda r(\theta))\nu(\theta)\mathrm{d}r(\theta) - \lambda\int_{-1}^{0} \exp(\lambda\tau(\theta))\eta(\theta)\mathrm{d}\tau(\theta). \tag{10}$$

Since $\nu(\theta)r'(\theta) \leqslant 0$ and $\eta(\theta)\tau'(\theta) \leqslant 0$ for $\theta \in [-1, 0]$, and $u\exp(-u) \leqslant 1/e$, for every real u, we have

$$M(\lambda) \geqslant \exp(-\lambda r(0))\nu(0) + \exp(\lambda\tau(0))\eta(0) + S_1.$$