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Finite element analysis for the axisymmetric Laplace operator on polygonal domains

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ABSTRACT

Let $\mathcal{L} := -r^{-2}(r\partial_r)^2 - \partial_z^2$. We consider the equation $\mathcal{L}u = f$ on a bounded polygonal domain with suitable boundary conditions, derived from the three-dimensional axisymmetric Poisson's equation. We establish the well-posedness, regularity, and Fredholm results in weighted Sobolev spaces, for possible singular solutions caused by the singular coefficient of the operator $\mathcal{L}$, as $r \to 0$, and by non-smooth points on the boundary of the domain. In particular, our estimates show that there is no loss of regularity of the solution in these weighted Sobolev spaces. Besides, by analyzing the convergence property of the finite element solution, we provide a construction of improved graded meshes, such that the *quasi-optimal* convergence rate can be recovered on piecewise linear functions for singular solutions. The introduction of a new projection operator from the weighted space to the finite element subspace, certain scaling arguments, and a calculation of the index of the Fredholm operator, together with our regularity results, are the ingredients of the finite element estimates.

1. Introduction

Let $\tilde{\Omega} := \Omega \times [0, 2\pi) \subset \mathbb{R}^3$ be a bounded domain, formed by the revolution of the polygon $\Omega \subset \mathbb{R}^2$ with respect to the z-axis (see Fig. 1). Consider the three-dimensional Poisson's equation in $\tilde{\Omega}$, with zero Dirichlet boundary conditions. In the presence of axisymmetry in the data, the Laplace operator in the three-dimensional domain becomes the two-dimensional elliptic operator

$$\mathcal{L} := -\frac{1}{r^2}(r\partial_r)^2 - \partial_z^2, \quad r > 0,$$

where r and z are the variables in the cylindrical coordinates (r, θ, z). Consequently, the three-dimensional axisymmetric Poisson's equation can be reduced to

$$\mathcal{L}u = f \quad \text{in } \Omega, \qquad u|_{\Gamma_0} = 0, \tag{1}$$

where $\Gamma_0 := \partial\tilde{\Omega} \cap \partial\Omega$. We are interested in studying the finite element method (FEM) for the elliptic equation (1). The reduction of the dimension (from three dimensions to two dimensions) leads to substantial savings on the computation of the numerical solution for the original three-dimensional elliptic boundary value problem, and hence is of practical interest.

Suppose the closure of the domain Ω intersects the z-axis. Despite the benefit in numerical computation, this process, however, introduces singular coefficients in the elliptic operator $\mathcal{L}$ and results in Sobolev spaces

$$H_r^m(\Omega) = \{v,\ r^{1/2}\partial_r^i\partial_z^j v \in L^2(\Omega),\ i+j \le m\}$$

with weights vanishing at $r = 0$, which raises difficulties both in the analysis of the equation and in the estimates of the FEM. For the validation on the reduction of the dimension, it is shown in [1,2] that, the three-dimensional Poisson's equation

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is equivalent to the two-dimensional equation (1), by using Fourier analysis to prove certain isomorphisms between the usual Sobolev spaces $H^m(\tilde{\Omega})$ and the weighted spaces $H_r^m(\Omega)$. An approximation property of the finite element solution for the axisymmetric Stokes problem in the space H_r^m is discussed in [3]. We also mention [4,5], in which the Fourier–FEM, a combination of the approximating Fourier and the FEM, is studied for the axisymmetric Poisson's equation. In addition, estimates on the convergence of the multigrid method for the axisymmetric Laplace operator and for the Maxwell operator can be found in [6,7], respectively.

Assuming sufficient regularity of the solution of Eq. (1), the existing results (see [3,6–8] and references therein) suggest that the H_r^1-norm of the error between the linear finite element solution and the real solution is bounded by Ch on the triangulation with quasi-uniform triangles of size h. This provides the analogy of the *quasi-optimal* convergence rate of the finite element solution for elliptic boundary value problems with regular coefficients in the usual Sobolev spaces and ensures good finite element approximations for the three-dimensional axisymmetric equation with a much lighter computational load than solving the original three-dimensional problem.

Furthermore, the solution of Eq. (1) may have singularities even in these weighted spaces $H_r^m(\Omega)$, due to the non-smooth points on the boundary $\partial\Omega$ and to the singular coefficient when $r \to 0$. The less regularity in the solution slows down the convergence rate of the finite element solution, as well as raises well-posedness concerns in these weighted spaces. Note that near the vertices of Ω that are not on the z-axis, the coefficients of the operator $\mathcal{L}$ are bounded and therefore, the singularities in the solution have the same character as the corner singularities of regular elliptic equations on polygonal domains. There exists a great deal of literature regarding different aspects of corner singularities of two-dimensional elliptic equations. See for example the monographs [9–14], research papers [15–24] on the analysis of the singular solution, and [25–28,16,29–31] and references therein on the numerical approximation for singular solutions of this type. For vertices on the z-axis, the situation is different, since the coefficient $1/r \to \infty$. It turns out that the possible singularities near these vertices are closely related to the three-dimensional vertex singularities of elliptic equations. This is our starting point for the work presented in this paper. See [32–36] for discussions on singular solutions of three-dimensional differential equations.

Different from the existing results mentioned above [3,2,6,7,4,5], we shall focus here on establishing well-posedness and regularity results for singular solutions of Eq. (1) in suitable Sobolev spaces and on the construction of simple, explicit finite element schemes to approximate these solutions quasi-optimally. Our goal shall be achieved by introducing the framework in a modified weighted Sobolev space $\mathcal{K}_{a,r}^m(\Omega)$ (Definition 2.7), which allows us to apply certain usual finite element formulations to Eq. (1). In the convergence analysis of the finite element solution, we introduce a new interpolation operator from a local regularization process (Definition 4.4). Compared with the usual nodal interpolation, this regularization technique demonstrates critical properties of functions in the weighted spaces, which are also useful to treat other axisymmetric problems (see [6,37]).

The rest of the paper is organized as follows. In Section 2, we first briefly recall some existing results in the literature for the axisymmetric equation. Then, we define two types of weighed Sobolev spaces for further analysis in Sections 3 and 4, as well as notation that will be used throughout this paper. In addition, several relevant properties of the weighted Sobolev space will be discussed.

In Section 3, we establish our a priori estimates (well-posedness, regularity, and the Fredholm property) for the axisymmetric equation in the weighted space $\mathcal{K}_{a,r}^m(\Omega)$. In particular, we shall show the operator

$$\mathcal{L} : \mathcal{K}_{a+1,+}^2(\Omega) \cap \{v|_{\Gamma_0} = 0\} \to \mathcal{K}_{a-1,r}^0(\Omega)$$

defines an isomorphism for $a > 0$ small and is Fredholm as long as a is away from a countable set of values. This allows us to compute the range of the index a, in which the isomorphism above still holds.

The finite element solution for Eq. (1) is studied in Section 4. In the first part of this section, we briefly present the approximation property of piecewise linear polynomials in the weighted space $H_r^2(\Omega)$. With a new interpolation operator, we show that the *quasi-optimal* convergence rate of the linear finite element solution is attained, assuming the solution is sufficiently regular. Based on these results and on a scaling argument, in the second part of Section 4, we analyze the convergence rate of the numerical solution in the weighted space $\mathcal{K}_{a,r}^m(\Omega)$. Then, we describe a construction of a sequence of triangulations suitably graded to the vertices, such that the *quasi-optimal* rate is recovered for singular solutions.

In Section 5, we present numerical tests for Eq. (1) on two domains for different singularities (on the z-axis or away from the z-axis). The rates of convergence of the finite element solutions from different meshes are compared. These tests suggest that the *quasi-optimal* convergence rates are achieved on our graded meshes, which is in complete agreement with the theory.

2. Weighted Sobolev spaces H_r^m and $\mathcal{K}_{a,r}^m$

In this section, we formally introduce the axisymmetric Poisson's equation and the definitions of some weighted Sobolev spaces with relevant properties.

2.1. The axisymmetric Poisson's equation

Let $\tilde{\Omega} := \Omega \times [0, 2\pi) \subset \mathbb{R}^3$ be a bounded domain, which is the revolution of Ω about the z-axis. Suppose $\tilde{\Omega}$ intersects the z-axis and its half section (the intersection of $\tilde{\Omega}$ and a meridian half plane) $\Omega \subset \mathbb{R}^2$ is a polygon (see, for example, Fig. 1).