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Conditions for coincidence of two cubic Bézier curves

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ABSTRACT

This paper presents a necessary and sufficient condition for judging whether two cubic Bézier curves are coincident: two cubic Bézier curves whose control points are not collinear are coincident if and only if their corresponding control points are coincident or one curve is the reversal of the other curve. However, this is not true for degree higher than 3. This paper provides a set of counterexamples of degree 4.

1. Introduction

Curve–curve intersection calculation is a basic problem in computer aided geometric design [1]. Sederberg and Meyer [2] proposed using bounding-box subdivision and a bounding wedge to calculate the intersection of two curves. Their method could be used to determine the intersection points of two curves, efficiently. However, when the two curves are coincident, their method will introduce infinite subdivisions. In fact, most of the intersection methods are not appropriate for the case where two curves are coincident [3–5]. A robust CAD system should have the ability to judge whether the two curves are coincident before making the intersection calculation.

Hu et al. [6] pointed out that if parts of two C^∞ regular curves are coincident, the two curves will not be separated at any points. This conclusion could be used to find the start point and the end point of the coincident part. However, it cannot be used to judge whether two curves are coincident.

An alternative way of judging whether two curves are coincident is to sample enough points on one curve, and judge whether these points are on the other curve. Garcia and Li [7] pointed out that the number of solutions of an equation system is $\Pi_{i=1}^{n} p_i$. Here n is the number of equations and p_i is the degree of the ith equation. Therefore, if two curves have enough common points, they are coincident. However, determining whether a point is on a parametric curve is not a trivial problem and is time-consuming.

Cubic Bézier curves are widely used in CAD systems. This paper proposes a necessary and sufficient condition for judging whether two cubic Bézier curves are coincident. It takes only a small number of computations to judge whether two such curves are coincident.

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2. The necessary and sufficient condition

Let $\mathbf{A}(t) = \sum_{i=0}^{3} B_{i,3}(t)\mathbf{P}_i$ and $\mathbf{B}(s) = \sum_{i=0}^{3} B_{i,3}(s)\mathbf{Q}_i$ be the two cubic Bézier curves. When the four control points of $\mathbf{A}(t)$ are collinear, the control points of $\mathbf{B}(s)$ should be collinear. If $\mathbf{P}_1$ and $\mathbf{P}_2$ are between $\mathbf{P}_0$ and $\mathbf{P}_3$, and $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are between $\mathbf{Q}_0$ and $\mathbf{Q}_3$, it is obvious that the two curves are coincident if and only if $\mathbf{P}_0 = \mathbf{Q}_0$ and $\mathbf{P}_3 = \mathbf{Q}_3$ or $\mathbf{P}_0 = \mathbf{Q}_3$ and $\mathbf{P}_3 = \mathbf{Q}_0$. Otherwise, we transform the coordinate of the control points so that all of them locate on the **X**-axis. Let x^A_{min} and x^A_{max} denote the minimum and maximum x-values of $\mathbf{A}(t)$, and x^B_{min} and x^B_{max} denote the minimum and maximum x-values of $\mathbf{B}(s)$. It is obvious that $\mathbf{A}(t)$ and $\mathbf{B}(s)$ are coincident if and only if $x^A_{min} = x^B_{min}$ and $x^A_{max} = x^B_{max}$.

Then we assume that the four control points of $\mathbf{A}(t)$ are not collinear. Without loss of generality, we set $\mathbf{P}_0$ as the coordinate origin. Since $\mathbf{A}(t)$ and $\mathbf{B}(s)$ are coincident, for any given $t \in [0, 1]$, there is an $s = s(t) \in [0, 1]$ satisfying that $\mathbf{A}(t) = \mathbf{B}(s(t))$. Also, for any given $s \in [0, 1]$, there is a $t = t(s) \in [0, 1]$ satisfying that $\mathbf{B}(s) = \mathbf{A}(t(s))$. We rewrite this relationship in a polynomial form:

$$a_{3x}t^3 + a_{2x}t^2 + a_{1x}t = b_{3x}s^3 + b_{2x}s^2 + b_{1x}s + b_{0x} \tag{1}$$

$$a_{3y}t^3 + a_{2y}t^2 + a_{1y}t = b_{3y}s^3 + b_{2y}s^2 + b_{1y}s + b_{0y} \tag{2}$$

$$a_{3z}t^3 + a_{2z}t^2 + a_{1z}t = b_{3z}s^3 + b_{2z}s^2 + b_{1z}s + b_{0z}. \tag{3}$$

Here $t, s \in [0, 1]$.

Lemma 2.1. *The equation shown in Eq. (4) can be derived from Eqs. (1)–(3), where* $a_2^2 + a_1^2 \neq 0$*:*

$$a_2t^2 + a_1t = b_3s^3 + b_2s^2 + b_1s + b_0. \tag{4}$$

Proof. Consider the coefficients a_{3x}, a_{3y} and a_{3z}.

1. If two or three of them are equal to 0, without loss of generality, we assume that $a_{3x} = a_{3y} = 0$. Then a_{2x}, a_{1x}, a_{2y} and a_{1y} cannot be equal to 0 simultaneously, because the control points of $\mathbf{A}(t)$ are not collinear. Therefore, Eq. (1) or Eq. (2) has the form of Eq. (4).
2. If two of them are not equal to 0, without loss of generality, we assume that $a_{3x} \neq 0$ and $a_{3y} \neq 0$. Eq. (1) multiplied by a_{3y}, minus Eq. (2) multiplied by a_{3x} yields

$$\hat{a_2}t^2 + \hat{a_1}t = \hat{b_3}s^3 + \hat{b_2}s^2 + \hat{b_1}s + \hat{b_0}.$$

   (a) If $a_{3z} \neq 0$, Eq. (1) multiplied by a_{3z}, minus Eq. (3) multiplied by a_{3x} yields a similar equation, whose coefficients of t^2 and t are denoted as $\bar{a_2}$ and $\bar{a_1}$, respectively. Since $\mathbf{P}_i$ $(i = 0, 1, 2, 3)$ are not collinear, $\hat{a_2}$, $\hat{a_1}$, $\bar{a_2}$ and $\bar{a_1}$ cannot be equal to 0 simultaneously; an equation with the form of Eq. (4) is obtained.
   (b) If $a_{3z} = 0$ and $a_{2z}^2 + a_{1z}^2 \neq 0$, Eq. (3) has the form of Eq. (4). Otherwise, $a_{2z} = a_{1z} = 0$. Then $\hat{a_2} \neq 0$ or $\hat{a_1} \neq 0$ because $\mathbf{P}_i$ $(i = 0, 1, 2, 3)$ are not collinear. Therefore, an equation with the form of Eq. (4) is obtained. □

Now we consider the relationship between $\mathbf{P}_i$ and $\mathbf{Q}_i$ by analyzing Eq. (4).

1. If $a_2 = 0$ and $a_1 \neq 0$, Eq. (4) could be expressed as

$$t = k_3s^3 + k_2s^2 + k_1s + k_0. \tag{5}$$

   (a) If $k_3 = k_2 = 0$, then $t = k_1s + k_0$ and $s = \frac{1}{k_1}(t - k_0)$.
      i. If $k_1 > 0$, the ranges of t and s are $[k_0, k_1 + k_0]$ and $[-\frac{k_0}{k_1}, \frac{1}{k_1}(1 - k_0)]$, respectively. Since t and s can be any values in the interval $[0, 1]$, we have $[0, 1] \subseteq [k_0, k_1 + k_0]$ and $[0, 1] \subseteq [-\frac{k_0}{k_1}, \frac{1}{k_1}(1 - k_0)]$. Therefore, $k_0 \leq 0$, $k_1 + k_0 \geq 1$ and $-\frac{k_0}{k_1} \leq 0$, $\frac{1}{k_1}(1 - k_0) \geq 1$. From the first two inequalities we obtain $1 - k_1 \leq k_0 \leq 0$, and $0 \leq k_0 \leq 1 - k_1$ is derived from the last two inequalities. Therefore, $k_1 = 1$, $k_0 = 0$ and $t = s$. In this case, $\mathbf{P}_i = \mathbf{Q}_i$ $(i = 0, 1, 2, 3)$.
      ii. If $k_1 < 0$, similarly, we can obtain $k_1 = -1$, $k_0 = 1$ and $t = 1 - s$. In this case, $\mathbf{P}_i = \mathbf{Q}_{3-i}$ $(i = 0, 1, 2, 3)$. $\mathbf{A}(t)$ is the reversal of $\mathbf{B}(s)$.
   (b) If $k_3 \neq 0$ or $k_2 \neq 0$, substitute Eq. (5) into Eq. (1) and let $a(s)$ be the corresponding polynomial. $b_{3x}s^3 + b_{2x}s^2 + b_{1x}s + b_{0x}$ is denoted as $b(s)$. Since $a(s) = b(s)$ for any $s \in [0, 1]$, we have $a(s) \equiv b(s)$. Therefore, the coefficient of $s^n (n > 3)$ in $a(s)$ is equal to 0. If $k_3 \neq 0$, the coefficient of s^9 in $a(s)$ is $a_{3x}k_3^3$, which means that $a_{3x} = 0$. Otherwise, $k_3 = 0$ and $k_2 \neq 0$. The coefficient of s^6 is $a_{3x}k_2^3$. We also have $a_{3x} = 0$. $a_{3y} = a_{3z} = 0$ can be obtained in the same way. Like for Lemma 2.1, we can prove that there is an equation with the following form:

$$t = l_3s^3 + l_2s^2 + l_1s + l_0.$$