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A simpler analysis of a hybrid numerical method for time-dependent convection–diffusion problems

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ABSTRACT

A finite difference method for a time-dependent convection–diffusion problem in one space dimension is constructed using a Shishkin mesh. In two recent papers (Clavero et al. (2005) [2] and Mukherjee and Natesan (2009) [3]), this method has been shown to be convergent, uniformly in the small diffusion parameter, using somewhat elaborate analytical techniques and under a certain mesh restriction. In the present paper, a much simpler argument is used to prove a higher order of convergence (uniformly in the diffusion parameter) than in [2,3] and under a slightly less restrictive condition on the mesh.

1. Introduction and summary

Consider the singularly perturbed initial-boundary value problem

$$u_t + L_\varepsilon u = f \quad \text{on } \Omega := (0, 1) \times (0, T], \tag{1a}$$

$$u(x, 0) = s(x) \quad \text{for } 0 \le x \le 1, \tag{1b}$$

$$u(0, t) = u(1, t) = 0 \quad \text{for } 0 < t \le T, \tag{1c}$$

where

$$L_\varepsilon u(x, t) \equiv -\varepsilon u_{xx}(x, t) + a(x)u_x(x, t) + b(x, t)u(x, t), \tag{2}$$

with $a(x) > \alpha > 0$ and $b = b(x, t) \ge 0$ on $\bar{\Omega}$. The diffusion coefficient ε is a small positive parameter. Further assumptions on the data of the problem will be given in Section 2.

From (8) one sees that the solution u of (1) has an exponential boundary layer at the side $x = 1$ of Ω. Consequently, classical numerical methods on equidistant meshes do not give satisfactory results unless the mesh width depends on the value of the diffusion parameter ε and is small. Several numerical methods that yield accurate numerical solutions for (1), uniformly in ε, have been proposed in the literature; see [1, Part II].

In the present paper, we focus on two finite difference methods for (1) that were presented and analysed in recent papers in [2,3]. Convergence, uniformly in ε, is proved for these methods in these papers under the restriction that $b = b(x)$. Both papers use the same mesh (equidistant mesh in time with mesh spacing τ, piecewise-equidistant Shishkin mesh in space with N mesh intervals) and backward Euler differencing to approximate the time derivative, but their spatial discretizations seem to be different: Clavero et al. use the second-order HODIE scheme from [4] while Mukherjee and Natesan favour the hybrid difference scheme of [5]. In Section 3, we shall show that in fact the methods of [2,3] are essentially identical,

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despite the claim in [3, Introduction] that the method of [3] is simpler than that of [2]. We do this for the more general case $b = b(x, t)$.

We prove in Section 4 that these numerical methods are convergent, uniformly in ε, when applied to (1). Our proof is much simpler than the proofs in [2,3] as it avoids any asymptotic analysis of the semidiscrete problem that results from a discretization only in time. It operates under the mesh assumption (15), which is much less restrictive than the mesh restriction $N^{-q} \le C\tau$ for some $q \in (0, 1)$ that is assumed in [2,3]. (Here and subsequently C denotes a generic positive constant that is independent of ε and of the mesh.) When $\varepsilon \le N^{-1}$, our convergence result (Theorem 1) becomes

$$\max_{i,j} |u(x_i, t_j) - U_i^j| \le C[\tau + (N^{-1} \ln N)^2]. \tag{3}$$

This sharpens the weaker result

$$\max_{i,j} |u(x_i, t_j) - U_i^j| \le C[\tau + N^{-2+q}(\ln N)^2]$$

that was derived in [2,3]. The numerical results presented in [2,3] show that the factor N^q here is an artefact of the analysis, i.e., that our bound (3) is sharp. We give a further numerical example in Section 5 to illustrate that our convergence result is indeed sharp.

2. Assumptions on the data

We assume that all data of the problem are smooth and that the following zero-order and first-order corner compatibility conditions are satisfied:

$$s(0) = s(1) = 0, \qquad -\varepsilon s''(0) + a(0)s'(0) = f(0, 0), \qquad -\varepsilon s''(1) + a(1)s'(1) = f(1, 0). \tag{4}$$

Then (1) has a unique solution in the parabolic Hölder space $C^{2+\alpha,1+\alpha/2}(\bar{\Omega})$ (see [6,1]). We also assume that the second-order corner compatibility conditions are satisfied so that $C^{4+\alpha,2+\alpha/2}(\bar{\Omega})$. These conditions can be written down explicitly in terms of the data of the problem in the following way: differentiating (1a) with respect to t yields

$$f_t = u_{tt} + L_\varepsilon u_t + b_t u = u_{tt} + L_\varepsilon(f - L_\varepsilon u) + b_t u.$$

Hence, recalling (1b), (1c) and (4), the second-order corner compatibility conditions are

$$L_\varepsilon(L_\varepsilon s) = L_\varepsilon f - f_t \quad \text{at the corners } (0, 0) \text{ and } (1, 0). \tag{5}$$

Under these hypotheses, the solution u of (1) has an exponential layer along the boundary $x = 1$ of Ω and satisfies the bound

$$\left| \frac{\partial^{k+m} u(x, t)}{\partial x^k \partial t^m} \right| \le C(1 + \varepsilon^{-k} e^{-\alpha(1-x)/\varepsilon}) \quad \text{for } (x, t) \in \bar{\Omega} \text{ and } k + 2m \le 4. \tag{6}$$

This can be shown using the techniques described in [1, Part II, Section 2.2].

The a priori inequality (6) is all that is needed for most of our analysis. In a single place – the derivation of (28) below – we also need this inequality when $k = 4$ and $m = 1$, which is not included in (6). To prove this additional bound, we are forced to assume also that the data of the problem (1) satisfy the third-order compatibility condition

$$f_{tt} = L_\varepsilon(f_t - L_\varepsilon(f - L_\varepsilon s) - b_t s) \quad \text{at the corners } (0, 0) \text{ and } (1, 0). \tag{7}$$

Then, similarly to the derivation of (6), one can show that

$$\left| \frac{\partial^{k+m} u(x, t)}{\partial x^k \partial t^m} \right| \le C(1 + \varepsilon^{-k} e^{-\alpha(1-x)/\varepsilon}) \quad \text{for } (x, t) \in \bar{\Omega} \text{ and } k + 2m \le 6. \tag{8}$$

This is in contrast to [2,3] who assume that (8) is valid for $k + m \le 4,\ m \le 2$.

Remark 1. Numerical results indicate that when (7) is violated, the rate of convergence of our numerical method is unaffected. See Section 5 for an example.

Remark 2. As ε can take a range of values, the compatibility condition (4) implies that

$$s(0) = s(1) = 0, \qquad a(0)s'(0) = f(0, 0), \qquad a(1)s'(1) = f(1, 0), \qquad s''(0) = s''(1) = 0. \tag{9}$$

Similarly, by invoking (9) one sees that (5) is equivalent to requiring

$$(a' + b)f = af_x - f_t, \qquad (a'' + 2b_x)s' = f_{xx}, \qquad s^{(4)} = 0$$

at the corners (0,0) and (1,0). The assumption (7) places further conditions on the data, though as Remark 1 indicates, these may not needed in practice.

Although these conditions restrict the permissible data, nevertheless it is clear that are satisfied by certain data—for example if sufficiently many derivatives of s and f vanish at the corners $(0, 0)$ and $(1, 0)$.