Journal of Computational and Applied Mathematics 236 (2011) 184–195

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A third order partial differential equation for isotropic boundary based triangular Bézier surface generation

A. Arnal [a,*], J. Monterde [b]

[a] Dep. de Matemàtiques, Universitat Jaume I, Castelló, Spain

[b] Dep. de Geometria i Topologia, Universitat de València, Burjassot (València), Spain

ARTICLE INFO

Article history:
Received 18 January 2011
Received in revised form 6 May 2011

Keywords:
Partial differential equation
PDE surface
Explicit solution
Isotropy
Bézier triangle

ABSTRACT

We approach surface design by solving a linear third order Partial Differential Equation (PDE). We present an explicit polynomial solution method for triangular Bézier PDE surface generation characterized by a boundary configuration. The third order PDE comes from a symmetric operator defined here to overcome the anisotropy drawback of any operator over triangular Bézier surfaces.

1. Introduction

The broad aim of this work is to develop a surface design technique for triangular Bézier surfaces based on the boundary information. Given a boundary, we create a surface as an explicit solution to an appropriately chosen PDE. In 1989, Bloor and Wilson gave these types of surface modeling techniques the name "PDE surfaces"; see [1]. Since most information defining a surface comes from its boundary curves, adding some boundary conditions to the PDE allows the PDE based method to generate and control the surface shape through very few parameters.

A triangular Bézier surface satisfying a linear PDE can be determined given some of its control points. The minimum set of prescribed control points depends on the PDE under study, mainly on its order. Previous work on this subject was performed in [2–5]. The cited papers study the generation of rectangular Bézier surfaces satisfying the Laplace equation as well as the biharmonic equation. Moreover, an analogous study about the Bézier solutions of the wave equation can be found in [6]. In the case of harmonic Bézier surfaces two boundary conditions were required to construct the surface while for biharmonic Bézier surfaces four boundary curves were needed as initial data. It is noteworthy to point out that we were indeed looking for polynomial solutions of the biharmonic equation (not for arbitrary functions). Recalling this, by prescribing the values along the boundary, but not for the tangent plane, a polynomial solution of the biharmonic equation is fixed.

More in general, in [7] a method was given for generating rectangular Bézier surfaces as a solution of a fourth order elliptic PDE, the Euler–Lagrange equation associated with the most general quadratic functional. Finally, in [8] an explicit polynomial solution of this general fourth order elliptic PDE was given.

For triangular Bézier surfaces in [9], we approached surface design by solving second order and fourth order PDEs. We presented many methods for designing triangular Bézier PDE surfaces given different sets of prescribed control points which include the special cases of harmonic and biharmonic surfaces. We saw that, given two lines of control points, a triangular

* Corresponding author. Tel.: +34 616831630; fax: +34 964 728429.
E-mail addresses: parnal@mat.uji.es (A. Arnal), monterde@uv.es (J. Monterde).
URL: http://www.uv.es/monterde/ (J. Monterde).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2011.06.017

Bézier surface is totally determined by the harmonic condition which is a second order PDE. In the biharmonic case, a fourth order PDE, we were able to determine a biharmonic triangular Bézier patch given four lines of control points.

Triangular Bézier surfaces have an advantage over rectangular Bézier surfaces when dealing with derivatives. Let us illustrate it with an example: Given a polynomial $u^n v^n$, as a tensor-product Bézier surface coordinate function, its degree is (n, n); but if it is considered as a triangular Bézier coordinate function, its degree would be $2n$. Then if we take derivatives, for example $(\frac{\partial}{\partial u} + \frac{\partial}{\partial v})(u^n v^n) = n(u^{n-1}v^n + u^n v^{n-1})$, in terms of rectangular patches, its degree is still (n, n), but it decreases to $2n - 1$ in terms of triangular Bézier surfaces. Therefore we can consider that triangular Bézier patches are better adapted to PDE problems.

It is then natural to think about our next goal: Given all the boundary control points (three lines), to determine an associated Bézier surface that will fulfill a third order condition. In this way, we introduce a third order linear PDE with constant coefficients and we give an explicit formula to generate the associated PDE surface from a prescribed boundary.

The harmonic and biharmonic operators are isotropic for functions defined in $\mathbb{R}^2$, but if we consider triangular Bézier surfaces we find an anisotropic behavior due to the fact that one of the three barycentric coordinates must be eliminated before applying the operator. The directions of the coordinate axes in the parameter domain take a special role, because triangular Bézier surfaces are harmonic or biharmonic only for a certain labeling of the axes: the domain triangle is embedded into the plane $\mathbb{R}^2$ by an affine transformation, where one of the three angles becomes a right angle and the two edges meeting there get length 1. Since there are three possibilities for choosing this angle, the result depends on this choice; if one swaps two of the three indices of the control points then the harmonic or biharmonic property may be destroyed.

Therefore, to solve the anisotropy drawback of these operators over Bézier triangles we introduced in [9] a second order and a fourth order differential operator, isotropic with respect to triangular Bézier parameterizations. That is, an operator whose action over a triangular Bézier parametrization remains invariant under permutation of the indices of the control net; see Eq. (7). Now, we also look for an isotropic third order differential operator to define the third order linear PDE that will determine a surface from its boundary. Moreover we see that applying symmetry criteria on the mask associated to a general third order linear PDE with constant coefficients leads to isotropy too.

The scheme of this paper is the following. In Section 2 we define the isotropic third order operator, introduce the general third order linear PDE and discuss its mask formulation in terms of different symmetry conditions.

In Section 3 we prove the existence of solutions to our problem. In Section 4 we compute an explicit polynomial solution to the third order linear PDE under our study. In Section 5 we expose that a PDE surface satisfying this PDE can be determined from a prescribed boundary. We give some examples of Bézier solutions and compare them with the harmonic and biharmonic surfaces we obtained in [9].

In Section 6 we discuss the relation of our PDE surfaces to the permanence principle; see [10]. Finally in Section 7 we present some conclusions.

2. The third order PDE

The mathematical models of science and engineering usually take the form of differential equations, in many cases differential equations under study (harmonic, biharmonic or waves equation for example) arise naturally from a general physical principle. However, since up to our knowledge, there is no third order PDE coming from a model problem, let us start with a general third order linear operator with constant coefficients:

$$R\overrightarrow{x} = \alpha \overrightarrow{x}_{uuu} + \beta \overrightarrow{x}_{uuv} + \gamma \overrightarrow{x}_{vvu} + \delta \overrightarrow{x}_{vvv} \tag{1}$$

and the corresponding third order PDE: $R\overrightarrow{x} = 0$.

Let $\overrightarrow{x}(u, v) = \overrightarrow{x}(u, v, 1 - u - v) = \sum_{|I|=n} P_I B_I^n(u, v)$ be a triangular Bézier surface with control net $\{P_I\}_{|I|=n}$, where $B_I^n(u, v)$ are the bivariate Bernstein polynomials $B_I^n(u, v) = \binom{n}{I} u^i v^j (1 - u - v)^k$, with $I = \{i, j, k\}$ and $|I| = n$.

The third order linear PDE

$$\alpha \overrightarrow{x}_{uuu} + \beta \overrightarrow{x}_{uuv} + \gamma \overrightarrow{x}_{vvu} + \delta \overrightarrow{x}_{vvv} = 0, \tag{2}$$

in terms of control points is

$$\begin{aligned} 0 &= \left(\alpha \frac{\partial^3}{\partial u^3} + \beta \frac{\partial^3}{\partial u^2 v} + \gamma \frac{\partial^3}{\partial v^2 u} + \delta \frac{\partial^3}{\partial v^3}\right) \overrightarrow{x}(u, v) \\ &= n(n-1)(n-2) \sum_{|I|=n-3} (\alpha \Delta^{3,0} P_I + \beta \Delta^{2,1} P_I + \gamma \Delta^{1,2} P_I + \delta \Delta^{0,3} P_I) B_I^{n-3}(u, v) \\ &= \sum_{|I|=n-3} Q_I \, B_I^{n-3}(u, v) \end{aligned}$$

where $\Delta^{l,m}$ denote the differences

$$\begin{aligned} \Delta^{l,m} P_{i,j,k} &= \Delta^{l-1,m} \Delta^{1,0} P_{i,j,k} = \Delta^{l-1,m}(P_{i+1,j,k} - P_{i,j,k+1}), \\ \Delta^{l,m} P_{i,j,k} &= \Delta^{l,m-1} \Delta^{0,1} P_{i,j,k} = \Delta^{l,m-1}(P_{i,j+1,k} - P_{i,j,k+1}). \end{aligned}$$