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Symmetry analysis of a model of stochastic volatility with time-dependent parameters

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ABSTRACT

We provide the solutions for the Heston model of stochastic volatility when the parameters of the model are constant and when they are functions of time. In the former case, the solution follows immediately from the determination of the Lie point symmetries of the governing 1 + 1 evolution partial differential equation. This is not the situation in the latter case, but we are able to infer the essential structure of the required nonlocal symmetry from that of the autonomous problem and hence can present the solution to the nonautonomous problem. As in the case of the standard Black–Scholes problem the presence of time-dependent parameters is not a hindrance to the demonstration of a solution.

1. Introduction

Recently, Sophocleous et al. [1] provided a solution of the Stein–Stein model for stochastic volatility [2] in terms of an algorithmic process based upon the Lie Theory of infinitesimal transformations and its associated group theory. The solution was provided in two instances. The first was the autonomous problem presented in [3] and the second was a nonautonomous version of the same problem introduced in [4].

In both cases the symmetry analysis showed that the algebraic structure of the evolution partial differential equation of the model,

$$2u_t + \beta^2 u_{xx} - \beta^2 \left(1-\rho^2\right) u_x^2 + 2\left(m - (\alpha + \xi\beta\rho)\, x\right) u_x + \xi^2 x^2 = 0, \tag{1.1}$$

where the parameters, apart from m, could depend upon time, was independent of the nature of the functions of time in the coefficients (apart from the natural properties of differentiability to the necessary orders required by the analysis) provided that ρ was a constant. When coupled with the terminal conditions[1] $u(T, x) = 0$, there were two symmetries remaining. As (1.1) possessed the maximal number of Lie point symmetries, one of the symmetries was a combination

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[1] Although this looks like a single condition, in terms of the symmetry analysis it is two since the variables, t, x and u, are treated as independent variables. Thus the condition mentioned in the full text is in fact the dual condition, $t = T$ and $u = 0$ for all values of x.

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of the symmetries associated with the Weyl–Heisenberg subalgebra of the full symmetry group of (1.1) and the second a combination associated with the $sl(2, R)$ subalgebra. This is not an unusual situation in the case of evolution partial differential equations of maximal or near-maximal symmetry when it comes to problems in Financial Mathematics [5–11].

When ρ was not a constant, *i.e.* the coefficients of u_{xx} and u_x^2 were not constantly proportional, there was a considerable reduction in the number of Lie point symmetries. The infinite subalgebra, indicating that the equation was in fact a linear equation in disguise, disappeared. Also two elements of the previously existing Weyl–Heisenberg subalgebra disappeared. The single remaining symmetry, ∂_u, obvious from the absence of u in (1.1) of the Weyl–Heisenberg subalgebra, and the three elements of $sl(2, R)$ remained provided that there was a constraint between the coefficients of the equation. The constraint did not have the simplicity of ρ being a constant! As it happened, the need for the constraint disappeared when one applied the terminal condition. The remaining three symmetries were sufficient to provide a similarity solution of (1.1) subject to the terminal condition.

The richness of the results resulting from the application of symmetry methods to the Stein–Stein model of stochastic volatility prompts one to look at another model, proposed in [12]. Using a standard arbitrage argument, u must satisfy the following PDE

$$2u_t + \beta^2 u_{xx} - \beta^2\left(1-\rho^2\right)u_x^2 + 2\left(\frac{m}{x} - (\alpha + \xi\beta\rho)\,x\right)u_x + \xi^2 x^2 = 0. \tag{1.2}$$

The terminal condition remains as $u(T, x) = 0$. In [3] α, β, ρ and ξ are taken as constants whereas Kufakunesu [4] takes them to have an explicit dependence upon the time.

Our approach to the analysis of (1.2) and the associated terminal condition is based upon the Lie algebraic analysis of the equation to see if there exists a sufficient number of symmetries so that there is the possibility of the existence of a symmetry of the equation which is compatible with the terminal conditions, $u = 0$ when $t = T$ for all x. We observe that this approach has been successful in a number of analyses of evolution partial differential equations which arise in Financial Mathematics; see for example [7,6,9,13,8,10,14,11,5]. As the calculation of the Lie symmetries of a differential equation is usually a tediously nonintellectual activity, we make use of one of the symbolic manipulation packages available for the purpose. Our choice is Sym [15–17], but there are several other packages which should be equally effective. In view of the number of parameters in (1.2), be they constants or time-dependent functions, an interactive approach is necessary. For this Sym is well suited. The same is true of the other two packages of known robustness, those of Alan Head [18] and of Clara Nucci [19,20].

In Section 2 we analyse (1.2) as in the model of Benth and Karlsen for its Lie point symmetries and see how they can be applied to obtain the solution of the problem with the terminal conditions. We note that there is an interesting algebraic variation in the results. In Section 3 we make the analysis with the variation proposed by Kufakunesu. We see that there is a big difference in the analysis and that this constitutes one of the more important aspects of this paper. We conclude in Section 4 with some general comments and observations.

2. The Heston volatility model

The assumption of constant volatility in the classical Black–Scholes–Merton model is evidently inadequate to include the smile phenomena observed in the financial markets. Models of stochastic volatility have appeared in recent years to treat such behaviour. The model of stochastic volatility due to Heston is considered to be one of the most amenable to further analysis. One of the limitations is that the closed-form solution to the pricing formula may thus far only be derived when the associated parameters are constant [12] or piecewise constant [21]. This point was advanced in a recent work in [22] in which the authors use the methods of Malliavan calculus to establish an analytic formula for the pricing of European options for time-dependent parameters in the case that the volatility of volatility is relatively small. Here we find solutions to Heston's model for parameters which are constant or functions of time. We place no restrictions upon the parameters.

When we apply Sym in interactive mode to (1.2) we find that a symmetry has the form

$$\Gamma = a(t)\partial_t + \left(\frac{1}{2}\dot{a}x + b(t)\right)\partial_x + \left\{G(t,x) + \exp\left[\left(1-\rho^2\right)u\right]F(t,x)\right\}\partial_u, \tag{2.1}$$

where $F(t, x)$ is a solution of

$$2F_t + \beta^2 F_{xx} + 2\left[\frac{m}{x} - (\alpha + \beta\xi\rho)x\right]F_x - \left(1-\rho^2\right)\xi^2 x^2 F = 0, \tag{2.2}$$

which means that there exists a linearising transformation for (1.2), and

$$G(t,x) = g(t) - \frac{1}{4\beta^2\left(1-\rho^2\right)}\left[(\ddot{a} + 2(\alpha+\beta\xi\rho)\dot{a})\,x^2 + 4\left((\alpha+\beta\xi\rho)b(t) + \dot{b}\right)x - 4\frac{mb(t)}{x}\right]. \tag{2.3}$$