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A general algorithm for the numerical evaluation of nearly singular integrals on 3D boundary element

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ABSTRACT

A general numerical method is proposed to compute nearly singular integrals arising in the boundary integral equations (BIEs). The method provides a new implementation of the conventional distance transformation technique to make the result stable and accurate no matter where the projection point is located. The distance functions are redefined in two local coordinate systems. A new system denoted as (α, β) is introduced here firstly. Its implementation is simpler than that of the polar system and it also performs efficiently. Then a new distance transformation is developed to remove or weaken the near singularities. To perform integration on irregular elements, an adaptive integration scheme is applied. Numerical examples are presented for both planar and curved surface elements. The results demonstrate that our method can provide accurate results even when the source point is very close to the integration element, and can keep reasonable accuracy on very irregular elements. Furthermore, the accuracy of our method is much less sensitive to the position of the projection point than the conventional method.

1. Introduction

Accurate evaluation of boundary integrals with various kernel functions of the type $O(1/r^{\chi})$ is an important issue in the implementation of the boundary type numerical methods based on the boundary integral equations (BIEs), such as the boundary element method (BEM), the boundary face method (BFM) [1]. r is the distance between the source point and the field point. These integrals become singular or nearly singular when the source point collides with or is close to the field point. The conventional Gaussian quadrature becomes inefficient or even inaccurate to evaluate these integrals. Special integration techniques are urgently needed to deal with these integrals. In this work, we focus on numerical evaluation of nearly singular integrals in three dimensions.

The nearly singular integral arises in mainly five cases: (a) the concerned structure is thin [2,3]; (b) the neighboring element sizes of a surface are quite different [4]; (c) the element's shape is very irregular [1]; (d) the interior points are close to the boundary in the post-processing; (e) for crack problems. In most cases the number of the nearly singular integrals can be much larger than that of the singular integrals in computation of the system matrix, because the singular integrals are involved in the evaluation of the main diagonal entries only. Therefore, efficient and accurate evaluation of the nearly singular boundary integrals may be a key factor in the overall performance of the BEM or BFM [5].

To remove the near singularities, various methods have been proposed, such as the element subdivision technique [1,6], analytical and semi-analytical methods [7,8], non-linear transformation techniques [9–16] and distance transformation techniques [17–19]. The element subdivision method is accurate but inefficient, and may be instable when the distance

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is very small compared with the element size. The analytical and semi-analytical methods are effective, but are limited to the planar elements only. When curved elements are involved, these elements must be divided into a large number of planar triangles, thus losing efficiency and accuracy. Most variable transformation techniques are efficient. However, it is difficult to find a general method which is effective for a wide range of nearly singular integrals and can be used to compute nearly singular integrals on different boundary elements. Moreover, these methods are dependent on kernel functions, and complicated mathematical deductions for different kernels are required. Distance transformation method [17,18], which has been proposed by Ma et al., is a general strategy to deal with nearly singular integrals with various kernels in BEM. This promising method is derived from Guiggiani's excellent work for dealing with singular boundary integrals [19]. For this method, the numerical results are very sensitive to the position of the projection point of the source point. This is also the common drawback of the methods discussed above. Moreover, the projection point is defined in a rigorous way, namely, the line consisting of the source point and the projection point must be perpendicular to the tangential plane through the projection point for 3D boundary elements. According to the definition, the following two difficult cases may be encountered. If the source point is located inside the tangential plane through the projection point [17], the method fails when the transformation is performed based on the local Cartesian coordinate system. If the projection point is located outside the element, troubles may be introduced due to the complexity of determining the local variable ρ and θ intervals of the local poplar system. It is necessary to consider both the position of the projection point (e.g. outside or inside the element) and the element's shape (e.g. triangle or quadrangle).

In this paper, we present a new implementation of the distance transformation technique, and extend the technique to evaluate nearly singular integrals on parametric surface elements used in BFM [1,20]. In our implementation, the projection point of the source point is defined in a more general form. The troubles detailed above are circumvented using our method. What is more, our method has an attractive feature that its accuracy is much less sensitive to the position of the projection point. This performance makes our method is very practical in actual problems. We also introduce a new local coordinate system [1,20] described by (α, β) into the area of nearly singular integration firstly. This system is similar to the polar system, but its implementation is simpler than the polar system and it also performs efficiently. To deal with nearly singular integrals on slender surface elements, the element subdivision technique is employed here in combination with our method. Although element subdivision is adopted, the computational cost is reduced dramatically compared with the conventional subdivision techniques [1,6,20].

The goal of our work is to develop a general method that is suitable for various 3D boundary elements including planar and curved surface elements and very irregular elements with slender shape in physical space. In this paper, we deal with integrals with near weak and strong singularities appearing in BIEs, and the evaluation of nearly hypersingular integrals will be reported in a forthcoming paper.

2. Statement of the problem

In this paper, we deal with the following boundary integral with near singularity over 3D boundary element S:

$$I = \int_S \frac{f(\mathbf{y}, r)}{r^\chi} \phi(\mathbf{x}) \mathrm{d}S(\mathbf{x}), \quad \chi = 1, 2 \tag{1}$$

where $\mathbf{y}$ and $\mathbf{x}$ are referred to as the source point and the field point in BEM or BFM, respectively, $\mathbf{y}$ is very close to S, r is the Euclidean distance between $\mathbf{y}$ and $\mathbf{x}$, f is a well-behaved function, and $\phi(\mathbf{x})$ is a shape function. Since $\mathbf{y}$ is outside the integration element S but very close to it, the integrals (1) become nearly singular. This problem usually is referred to as the boundary layer effect in BEM.

Now, we consider the boundary integral equations of 3D potential problems in the domain Ω enclosed by the boundary Γ. The two basic functions are presented in terms of the flux q and potential u on the boundary as follows [17]:

$$c(\mathbf{y})u(\mathbf{y}) = \int_\Gamma q(\mathbf{x})u^*(\mathbf{x}, \mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma u(\mathbf{x})q^*(\mathbf{x}, \mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) \tag{2}$$

$$c(\mathbf{y})u_k(\mathbf{y}) = \int_\Gamma q(\mathbf{x})u_k^*(\mathbf{x}, \mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) - \int_\Gamma u(\mathbf{x})q_k^*(\mathbf{x}, \mathbf{y})\mathrm{d}\Gamma(\mathbf{x}) \tag{3}$$

where c is a coefficient depending on the smoothness of the boundary at the source point $\mathbf{y}$. $u^*(\mathbf{x}, \mathbf{y})$ is the fundamental solution for the 3D problem expressed as

$$u^*(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi} \frac{1}{r(\mathbf{x}, \mathbf{y})}. \tag{4}$$

And $q^*(\mathbf{x}, \mathbf{y})$, $u_k(\mathbf{x}, \mathbf{y})$ and $q_k^*(\mathbf{x}, \mathbf{y})$ are all the derived fundamental solutions

$$q^*(\mathbf{x}, \mathbf{y}) = \frac{\partial u^*(\mathbf{x}, \mathbf{y})}{\partial \mathbf{n}}, \qquad u_k^*(\mathbf{x}, \mathbf{y}) = \frac{\partial u^*(\mathbf{x}, \mathbf{y})}{\partial x_k} \qquad q_k^*(\mathbf{x}, \mathbf{y}) = \frac{\partial q^*(\mathbf{x}, \mathbf{y})}{\partial x_k} \tag{5}$$

where $\mathbf{n}$ is the unit outward normal direction to the boundary Γ, with components n_i, $i = 1, 2, 3$.