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Impulsive anti-periodic boundary value problem of first-order integro-differential equations

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ABSTRACT

This paper is concerned with the anti-periodic boundary value problem of first-order nonlinear impulsive integro-differential equations. We first establish a new comparison principle, and then obtain the existence of extremal solutions by upper–lower solution and monotone iterative techniques. Some examples are presented to illustrate the main results.

1. Introduction

In recent years, there has been a great deal of research into the equations of existence and uniqueness of solutions to boundary value problems for differential equations [1]. Meanwhile, impulsive differential equations have also attracted more and more attention [2–6] because it is an important tool to study practical problems of biology, engineering and physics. The periodic boundary value problems involving impulsive differential equations have been studied by many authors; see [7–15] and the references therein. The first-order integro-differential equation with periodic boundary value condition has also been considered by many authors; see [12,16–18]. However, in real problems, some problems come down to anti-periodic boundary value problems. As far as we know, the papers concerned with anti-periodic boundary value problems are few; see [19–24].

In this paper, we consider the following nonlinear problem for first-order integro-differential equation with impulse at fixed points

$$y'(t) = f(t, y(t), (Ty)(t), (Sy)(t)), \quad t \in J_0, \tag{1.1}$$
$$\Delta y(t_k) = I_k(y(t_k)), \quad k = 1, 2, \ldots, p, \tag{1.2}$$
$$y(0) = -y(T), \tag{1.3}$$

where $J = [0, T]$, $J_0 = J \setminus \{t_1, t_2, \ldots, t_p\}$, $0 < t_1 < t_2 < \cdots < t_p < T$, $f \in C(J \times R \times R \times R, R)$, $I_k \in C(R, R)$, $\Delta y(t_k) = y(t_k^+) - y(t_k^-)$. And $y(t_k^+)$, $y(t_k^-)$ denote the right and left limits of $y(t)$ at t_k, $k = 1, 2, \ldots, p$,

$$(Ty)(t) = \int_0^t k(t, s)y(s)\mathrm{d}s, \qquad (Sy)(t) = \int_0^T h(t, s)y(s)\mathrm{d}s,$$

$k \in C(D, R^+)$, $D = \{(t, s) \in J \times J : t \geq s\}$, $h \in C(J \times J, R^+)$.

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Problems (1.1) and (1.2) were considered in several papers with different types of boundary conditions. In [12,25], Chen and Sun studied problems (1.1) and (1.2) and a similar problem to (1.1) and (1.2), respectively, under the boundary condition $g(y(0), y(T)) = 0$. There, the authors established the existence of extremal solutions by the upper and lower solutions and the monotone iterative technique when g satisfied some monotonicity conditions. Those results are applicable in some important case such as the initial or the periodic case. However, they are not valid for anti-periodic boundary conditions, i.e. $y(0) = -y(T)$.

Just recently, Luo and Nieto [17] proved some new comparison principles for periodic boundary value problem and extended the earlier results. Encouraged by paper [17], in this paper we first establish a new comparison principle for anti-periodic boundary value problem and then obtain the existence of extremal solutions for Eqs. (1.1)–(1.3) by the upper–lower solution and monotone iterative techniques.

2. Preliminaries and lemmas

In this section, we first introduce some definitions in order to define the concept of solution for Eqs. (1.1)–(1.3). Let

$$\begin{aligned} PC(J) &= \{y : J \to R : y \text{ is continuous at } t \in J_0;\\ &\qquad y(0^+), y(T^-),\ y(t_k^+) \text{ and } y(t_k^-) \text{ exist and } y(t_k^-) = y(t_k), k = 1, \dots, p\},\\ PC^1(J) &= \{y \in PC(J) : y \text{ is continuously differentiable for any } t \in J_0;\\ &\qquad y'(0^+), y'(T^-) \text{ and } y'(t_k^+),\ y'(t_k^-) \text{ exist}, k = 1, \dots, p\}. \end{aligned}$$

$PC(J)$ and $PC^1(J)$ are Banach spaces with the norms

$$\|y\|_{PC(J)} = \sup\{|y(t)| : t \in J\}, \qquad \|y\|_{PC^1(J)} = \|y\|_{PC(J)} + \|y'\|_{PC(J)}.$$

We say that a function y is a solution for Eqs. (1.1)–(1.3) if $y \in PC^1(J)$ and satisfies Eqs. (1.1)–(1.3).

In order to obtain the existence of solution for Eqs. (1.1)–(1.3), we need the following key lemma.

Lemma 2.1. *Assume that $y \in PC^1(J)$ satisfies*

$$\begin{cases} y'(t) + My(t) + N(Ty)(t) + N_1(Sy)(t) \le 0, & t \in J_0,\\ \Delta y(t_k) \le -L_k y(t_k), & k = 1, 2, \dots, p,\\ y(0) \le 0, \end{cases} \tag{2.1}$$

where $M > 0$, $N, N_1 \ge 0$, $L_k < 1$, $k = 1, 2, \dots, p$, and

$$\int_0^T q(s)\mathrm{d}s \le \prod_{j=1}^{p}(1 - \bar{L}_j) \tag{2.2}$$

with $\bar{L}_k = \max\{L_k, 0\}$, $k = 1, 2, \dots, p$,

$$q(t) = N \int_0^t k(t, s)\mathrm{e}^{M(t-s)} \prod_{s<t_k<T}(1 - L_k)\mathrm{d}s + N_1 \int_0^T h(t, s)\mathrm{e}^{M(t-s)} \prod_{s<t_k<T}(1 - L_k)\mathrm{d}s,$$

then $y \le 0$.

Proof. Let $c_k = 1 - L_k$ and $x(t) = (\prod_{t<t_k<T} c_k^{-1})y(t)\mathrm{e}^{Mt}$, then we have

$$\begin{cases} x'(t) \le -\left(\prod_{t<t_k<T} c_k^{-1}\right)\left[N\int_0^t k(t, s)x(s)\mathrm{e}^{M(t-s)}\left(\prod_{s<t_k<T} c_k\right)\mathrm{d}s + N_1\int_0^T h(t, s)x(s)\mathrm{e}^{M(t-s)}\left(\prod_{s<t_k<T} c_k\right)\mathrm{d}s\right],\\ \quad t \in J_0,\\ x(t_k^+) \le c_k x(t_k), \quad k = 1, 2, \dots, p,\\ x(0) \le 0. \end{cases} \tag{2.3}$$

Obviously, the function y and x have the same sign.

Suppose the contrary, then there exists a $t^* \in J$ such that $x(t^*) > 0$. Let $x(t_*) = \min_{t\in[0,t^*]} x(t) = b$, then $b < 0$. Otherwise, it follows the first equation of (2.3) that $x'(t) \le 0$ on $[0, t^*] \bigcap J_0$ so x is non-increasing. Thus $x(t^*) \le x(0) \le 0$, which is a contradiction. Therefore, Eq. (2.3) becomes

$$\begin{cases} x'(t) \le -b\left(\prod_{t<t_k<T} c_k^{-1}\right)q(t), & t \in J_0,\\ x(t_k^+) \le c_k x(t_k), & k = 1, 2, \dots, p,\\ x(0) \le 0. \end{cases} \tag{2.4}$$