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Explicit local time-stepping methods for Maxwell's equations☆

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ABSTRACT

Explicit local time-stepping methods are derived for time dependent Maxwell equations in conducting and non-conducting media. By using smaller time steps precisely where smaller elements in the mesh are located, these methods overcome the bottleneck caused by local mesh refinement in explicit time integrators. When combined with a finite element discretisation in space with an essentially diagonal mass matrix, the resulting discrete time-marching schemes are fully explicit and thus inherently parallel. In a non-conducting source-free medium they also conserve a discrete energy, which provides a rigorous criterion for stability. Starting from the standard leap-frog scheme, local time-stepping methods of arbitrarily high accuracy are derived for non-conducting media. Numerical experiments with a discontinuous Galerkin discretisation in space validate the theory and illustrate the usefulness of the proposed time integration schemes.

1. Introduction

The need to simulate electromagnetic wave phenomena of increasing complexity drives the quest for more general and efficient numerical methods. The first and probably most popular method, the finite difference time domain (FDTD) scheme [1], is simple and efficient on structured (Cartesian) grids, but on oblique or curved boundaries and interfaces it suffers from the inaccurate representation of the solution (staircase approximation) [2]. Moreover, higher order FDTD methods are generally difficult to implement near interfaces and boundaries. In contrast, finite element methods (FEMs) can handle unstructured grids and complex geometry, regardless of the order of approximation. They also provide rigorous a posteriori error estimates which are useful for local adaptivity and error control.

Different finite element discretisations of Maxwell's equations are available, such as edge elements [3–5] and nodal elements [6,7]. Although edge elements may be the most satisfactory from a theoretical point of view [8], in particular near re-entrant corners, they are less attractive for time-dependent computations because the solution of a linear system is required at every time step. Indeed, in the case of triangular or tetrahedral edge elements, the entries of the diagonal matrix resulting from mass-lumping are not necessarily strictly positive [9]. Therefore, explicit time-stepping cannot be used in general. In contrast, standard (H^1-conforming) nodal elements naturally lead to a fully explicit scheme when mass-lumping is applied, but cannot correctly represent corner singularities in general [8].

Discontinuous Galerkin (DG) FEMs offer an attractive alternative to edge elements for the numerical solution of Maxwell's equations, in particular for time-dependent problems. Not only do they accommodate elements of various types and shapes, irregular non-matching grids, and even locally varying polynomial order, and hence offer greater flexibility in the mesh design; they also lead to a block-diagonal mass matrix, with block size equal to the number of degrees of freedom per element. Thus when a spatial DG discretisation is combined with explicit time integration, the resulting time marching scheme will be truly explicit and inherently parallel.

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For the time-dependent Maxwell equations in first-order hyperbolic form, various DG methods are available [10–13], which combine high order nodal elements with low-storage Runge–Kutta (RK) time integration. By using a strong-stability-preserving RK scheme instead, improved accuracy and a less stringent time-step restriction can be achieved [14]. For Maxwell's equations in second-order form, a symmetric interior penalty (IP) DG method was proposed in [15,16], which yields optimal a priori error estimates in the energy norm and in the L^2-norm. In a non-conducting source-free medium, it also conserves (a discrete version of) the energy.

In the presence of complex geometry, adaptivity and mesh refinement are certainly key for the efficient numerical solution of Maxwell's equations. However, locally refined meshes impose severe stability constraints on explicit time-stepping schemes, where the maximal time-step allowed by a CFL condition is dictated by the smallest elements in the mesh [17]. When mesh refinement is restricted to a small region, the use of implicit methods, or a very small time step in the entire computational domain, are very high a price to pay. To overcome this stability restriction, various local time-stepping schemes [18,19] were proposed, which use implicit time-stepping or explicit smaller time-steps only where the smallest elements in the mesh are located. However, straightforward interpolation or extrapolation from the coarse to the finer space–time grid, say, generally results in low accuracy and poor stability properties. By enforcing the conservation of energy, an important ingredient for stability, Collino and Joly proposed a second-order local time-stepping method for the wave equation [18] and for Maxwell's equations [20] in a non-conducting medium. Although their approach remains explicit inside the coarse and the fine mesh, it nevertheless requires at every time step the solution of a linear system at the interface between the two grids.

Since DG methods are inherently local, they are particularly well suited for the development of explicit local time-stepping schemes [21]. By combining the symplectic Störmer–Verlet method with a DG discretisation, Piperno derived a symplectic local time-stepping scheme for Maxwell's equations in a non-conducting medium [22], which is explicit and second-order accurate. In [23], Montseny et al. combined a similar recursive integrator with discontinuous hexahedral elements. Although hexahedral elements are very efficient, they can produce spurious modes [24] while automated grid generation only with such elements remains a non-trivial task. Starting from the standard leap-frog scheme, Diaz and Grote [25] devised an explicit energy conserving local time-stepping scheme of arbitrarily high accuracy for the homogeneous wave equation. Recently, Taube et al. [26] proposed an explicit local time-stepping method for Maxwell's equations by extending the so-called arbitrary high-order derivatives (ADER) DG approach to Maxwell's equations; there, the solution is expanded in Taylor series in time and then the Cauchy–Kovalevskaya procedure is used to replace the time derivatives in this series by space derivatives.

Here we derive explicit local time-stepping methods for Maxwell's equations in conducting or non-conducting media with source terms. The rest of the paper is organized as follows. In Section 2, we present the Maxwell equations in second-order form and briefly recall the symmetric IP–DG formulation from [15,16] Section 3. Starting from the well-known second-order leap-frog scheme, we then derive in Section 4 explicit second-order local time-stepping schemes both in a conducting and in a non-conducting medium. In a non-conducting medium, we show that (a discrete version of) the energy is conserved, which provides a rigorous criterion for numerical stability. By using the modified equation approach, we then derive in Section 5 explicit local time-stepping methods of arbitrarily high accuracy in a non-conducting medium. Finally in Section 6, we present several numerical experiments, which validate the theory and illustrate the usefulness of the proposed local time-stepping schemes.

2. The Maxwell equations

The evolution of a time-dependent electromagnetic field $\mathbf{E}(\mathbf{x}, t)$, $\mathbf{H}(\mathbf{x}, t)$ propagating through a linear isotropic medium is governed by Maxwell's equations:

$$\begin{aligned} \varepsilon \mathbf{E}_t &= \nabla \times \mathbf{H} - \sigma \mathbf{E} + \mathbf{j}, \\ \mu \mathbf{H}_t &= \nabla \times \mathbf{E}. \end{aligned}$$

Here, the coefficients μ, ε and σ denote the relative magnetic permeability, the relative electric permittivity and the conductivity of the medium, respectively. The source term $\mathbf{j}$ corresponds to the applied current density. By eliminating the magnetic field $\mathbf{H}$, Maxwell's equations reduce to a second-order vector wave equation for the electric field $\mathbf{E}$:

$$\varepsilon \mathbf{E}_{tt} + \sigma \mathbf{E}_t + \nabla \times (\mu^{-1} \nabla \times \mathbf{E}) = \mathbf{j}_t.$$

If the electric field is eliminated instead, one easily finds that the magnetic field $\mathbf{H}$ satisfies a similar vector wave equation. Thus, we consider the following model problem: find the (electric or magnetic) field $\mathbf{u}(\mathbf{x}, t)$ such that

$$\begin{aligned} &\varepsilon \mathbf{u}_{tt} + \sigma \mathbf{u}_t + \nabla \times (\mu^{-1} \nabla \times \mathbf{u}) = \mathbf{f} \quad \text{in } \Omega \times (0, T), \\ &\mathbf{n} \times \mathbf{u} = 0 \quad \text{on } \partial\Omega \times (0, T), \\ &\mathbf{u}(\cdot, 0) = \mathbf{u}_0 \quad \text{in } \Omega, \\ &\mathbf{u}_t(\cdot, 0) = \mathbf{v}_0 \quad \text{in } \Omega. \end{aligned} \tag{1}$$