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A numerical investigation of blow-up in reaction–diffusion problems with traveling heat sources

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A R T I C L E I N F O

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A B S T R A C T

This paper studies the numerical solution of a reaction–diffusion differential equation with traveling heat sources. According to the fact that the locations of heat sources are known, we add auxiliary mesh points exactly at heat sources and present a novel moving mesh algorithm for solving the problem. Several examples are provided to demonstrate the efficiency of the new moving mesh method, especially in the case of two or three traveling heat sources. Moreover, numerical results illustrate that the speed of the movement of the heat source is critical for blow-up when there is only one traveling heat source. For the case of two traveling heat sources, blow-up depends not only on the speed but also on the distance between the two traveling heat sources.

1. Introduction

We consider an important blow-up problem in [1,2], which gives the following partial differential equation (PDE) with moving heat sources,

$$u_t - u_{xx} = \sum_{i=0}^{q} \delta(x - s_i) F_i[u(s_i, t)], \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+, \; q = 0, 1, \tag{1}$$

with the boundary and initial conditions

$$u(x, t) \to 0 \quad \text{as } |x| \to \infty, \tag{2}$$

$$u(x, 0) = u_0(x), \quad -\infty < x < \infty. \tag{3}$$

Here $u(x)$ is the temperature in an infinitely long medium. The initial temperature $u_0(x)$ is continuous and bounded with $u_0(x) \to 0$ as $|x| \to \infty$. The two sources are located at s_0 and s_1 with $s_1(t) = s_0(t) + d$, where $d > 0$. Thus, both sources move with the same speed. The nonlinear source functions F_i, $i = 0, 1$, are smooth and satisfy:

$$F_0^{(k)}(u) \geq F_1^{(k)}(u) > 0, \quad k = 0, 1, 2, \text{ for } u \geq 0.$$

Eq. (1) is important and used to study the combustion theory [3]. However, there are mainly two difficulties in solving problem (1). One is the delta function on the right-hand side of (1), which leads the solution derivatives discontinuous. The discretization of (1), using either finite difference method or finite element method, may meet with difficulty when one of the mesh elements crosses the time-dependent curves $s_i(t)$, $i = 0, 1$. Various approaches have been used to solve these interface problems. The immersed interface method (IIM) developed by Leveque and co-workers achieves much success [4].

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The main idea is to incorporate the known jumps of the solution or of its derivatives into the finite difference scheme to obtain a modified scheme which is second-order accurate.

Another difficulty is the blow-up phenomenon will happen at some time T. It has been proved by Kirk and Olmstead [2] when the heat source speed is sufficiently slow. Thus, the speed of moving source $s_0'(t)$ is critical for blow-up when there is only one heat source. For the case of two heat sources, the distance d is another critical point [1]. Considerable efforts have been devoted for the nonlinear reaction–diffusion equations with singularities [5,6]. Since the singularity developed on relatively small time intervals or spatial length scales, it is essential to use an adaptive mesh in the numerical simulation. With respect to the fixed mesh method, moving mesh method achieves the significant improvements in the accuracy and efficiency [7–9]. Recently, MMPDEs (moving mesh PDEs) approach is developed in [10] and has been successfully applied in a few blow-up partial differential equations [11–16]. In [10], several satisfactory mesh equations called MMPDEs are presented. Among them, MMPDE4, MMPDE5 and MMPDE6 are popular to use.

In [15], MMPDE6 is used to compute the Eq. (1) with one heat source, $q = 0$. Based on the idea for solving delta functions on fixed mesh [4,17], Ma and his co-worker develop an accurate approximate scheme by constructing a smooth function using the information of jumps. And five different approximations are derived depending on the location of the heat source. Then an accurate moving mesh algorithm is developed.

In this paper, we present a new moving mesh method for the Eq. (1). We note that the blow-up point is always located at the heat source (or one of them) and the positions of heat sources are the functions of time which have been given. Motivated by these, we add auxiliary mesh points exactly at the heat sources in every temporal discretization level. There is no doubt about the auxiliary points may help to accurately capture the singularity. Moreover, we simplify the investigation cases of the discretization in [15] and are adequate to solve the problem (1) with two moving sources. However, unlike MMPDEs in [15], the new method can be viewed as a combination of moving mesh method and a special adaptive method. The critical auxiliary points, which are introduced in Section 2, only exist at the current temporal level and do not move into the next temporal level. Thus, the total mesh points at every temporal level are either the same as N (the number of the initial mesh points), or at most $N + 1$ for one heat source or $N + 2$ for two heat sources. To compare with h-adaptive method, the new method can keep the advantages of moving mesh method. The influences of the velocity of heat source and the distance between two heat sources in occurrence of blow-up are investigated by numerical experiments.

The paper is organized as follows. In Section 2, we first briefly review moving mesh methods. Then, our algorithm for Eq. (1) is derived. In Section 3, a number of numerical experiments are presented to illustrate the accuracy and efficiency of the algorithm. Finally, conclusions are given in Section 4.

2. Moving finite difference algorithm

In this section, we derive a moving finite difference algorithm for the Eq. (1). First, moving mesh methods will be reviewed briefly. Then, the finite difference discretization of the Eq. (1) will be discussed.

2.1. Moving mesh methods

The moving mesh methods are used to obtain a time-dependent grid that is fit to the solution. Let x and ξ denote the physical and the computational coordinates, respectively. Without loss of generality, we suppose the physical and the computational domains are both given by [0, 1]. Then a one-to-one coordinate transformation between them is denoted by

$$x = x(\xi, t),$$

with

$$x(0, t) = 0, \qquad x(1, t) = 1.$$

Given a uniform mesh on the computation domain $\xi_i = \frac{i}{N},\ i = 0, 1, \ldots, N$, the corresponding non-uniform mesh in x is

$$0 = x_0 < x_1(t) < x_2(t) < \cdots < x_{N-1}(t) < x_N = 1.$$

The equidistribution principle (EQ) of the mesh [10] can be expressed as

$$\int_0^{x(\xi,t)} M(x, t)\mathrm{d}x = \xi \int_0^1 M(x, t)\mathrm{d}x,$$

where the monitor function $M(x, t)$ chosen to be some measure of the solution error plays a key role in the moving mesh method. Differentiating the above equation with respect to ξ twice, we obtain an equivalent differential form,

$$\frac{\partial}{\partial \xi}\left(M(x(\xi, t), t)\frac{\partial}{\partial \xi}x(\xi, t)\right) = 0. \tag{4}$$

Several MMPDEs can be derived from Eq. (4), see [10]. The MMPDE6

$$\frac{\partial^2 \dot{x}}{\partial \xi^2} = -\frac{1}{\tau}\frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right), \tag{5}$$

will be employed in this work.