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Oscillatory criteria for third-order nonlinear difference equation with impulses[☆]

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ABSTRACT

In this paper, we study the oscillation of a third-order nonlinear difference equation with impulses. Some sufficient conditions for the oscillatory behavior of solutions of third-order impulsive nonlinear difference equation are obtained. Some known results in the literature are generalized and improved.

1. Introduction

Consider the third-order nonlinear impulsive difference equation

$$\begin{cases} \Delta^3 x(n) + p(n) f(x(n-\tau)) = 0, & n \neq n_k,\ k = 1, 2, 3, \dots \\ \Delta^i x(n_k) = g_{i,k}(\Delta^i x(n_k - 1)), & i = 0, 1, 2,\ k = 1, 2, 3, \dots \end{cases} \tag{1.1}$$

where $p(n) \geq 0,\ p(n) \not\equiv 0$, $0 < n_0 < n_1 < n_2 < \cdots < n_k < \cdots$ and $\lim_{k\to\infty} n_k = \infty$, $\tau \in \mathbb{N}$, $\Delta x(n) = x(n+1) - x(n)$. Here, we always assume that the following conditions hold:

(A) $f : \mathbb{R} \to \mathbb{R}$ is continuous, and there exists an $\varepsilon_0 > 0$ such that $f(u)/u \geq \varepsilon_0 > 0$ for $u \neq 0$;

(B) $g_{i,k}(u)$ is continuous for $u \in (-\infty, +\infty)$, and there exist positive numbers $a_{i,k},\ b_{i,k}$ such that $a_{i,k} \leq \frac{g_{i,k}(u)}{u} \leq b_{i,k}$ for $u \neq 0,\ i = 0, 1, 2,\ k = 1, 2, 3, \dots$.

It is well known that the theory of impulsive differential/difference equations not only is richer than the corresponding theory of differential/difference equations without impulses but also provides a more adequate mathematical model for numerous processes and phenomena studied in physics, biology, engineering, etc. There has been a significant development in the theory of impulsive differential equations in the past several years, as various interesting results have been reported (see [1–11] etc., and the references therein). However, the development of the theory of impulsive difference equations is comparatively slow due to numerous theoretical and technical difficulties caused by their peculiarities [12–15]. In particular, to the best of our knowledge, there is little research work on the oscillation of impulsive third-order difference equations.

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Recently, Peng [12] studied a class of second-order nonlinear difference equations with impulses and obtained some oscillation criteria. They also considered the delay difference equations of third-order $\Delta^3 x_{n-2} + f(n, x_n, x_{n-l}) = 0$ with impulses $\Delta^2 x_{n_k-1} = M_k(\Delta^2 x_{n_k-2})$. But only one oscillation result was reported.

Li et al. [13] investigated Eq. (1.1) when $f(x) = x$, $g_{i,k}(x) = d_i x$, with constants d_i, $i = 0, 1, 2$, and obtained corresponding sufficient conditions for oscillatory behavior.

The present paper is motivated by the work of [13]. Our purpose is to tackle some sufficient conditions for the oscillation of solutions of Eq. (1.1). We extend Li's work and correct a mistake in Theorem 2, [13].

Let

$$c_{i,k} = \max\{|a_{i,k} - 1|, |b_{i,k} - 1|\} \quad \text{for } i = 0, 1, 2,\ k = 1, 2, 3, \ldots$$

and

$$n^{(k)} = n \cdot (n-1) \cdots (n-k+1), \quad k \in \mathbb{N}.$$

Definition 1.1. By a solution of Eq. (1.1) we mean a real-valued sequence $\{x(n)\}$, defined on $\{n_0-\tau, n_0-\tau+1, n_0-\tau+2, \ldots\}$ which satisfies Eq. (1.1) for $n \geq n_0$.

Definition 1.2. A solution of Eq. (1.1) is said to be nonoscillatory if the solution is eventually positive or eventually negative. Otherwise, the solution is oscillatory.

In this paper, we assume $a_{i,k}$, $i = 0, 1, 2$, $k = 1, 2, \ldots$ satisfy that

$$(\mathrm{H}): (n_1 - n_0) + a_{i,1}(n_2 - n_1) + a_{i,1}a_{i,2}(n_3 - n_2) + \cdots + a_{i,1}a_{i,2} \cdots a_{i,m}(n_{m+1} - n_m) + \cdots = \infty.$$

This paper is organized as follows. In Section 2, we shall offer some lemmas, which will be used in Section 3 to prove our main theorems. To illustrate our results, some examples are given in Section 4.

2. Some lemmas

In order to prove our theorems, we need the following lemmas.

Lemma 2.1. *Assume that $x(n)$ is a solution of Eq. (1.1) and (H) is satisfied. Given $i \in \{1, 2\}$, if there exists $N \geq n_0$ such that $\Delta^{i+1}x(n) \geq 0$ (≤ 0), $\Delta^i x(n) > 0$ (<0) for $n \geq N$. Then $\Delta^{i-1}x(n) \geq 0$ (≤ 0) holds for sufficiently large n.*

Proof. We only prove the conclusion under the assumption that $\Delta^{i+1}x(n) \geq 0$, $\Delta^i x(n) > 0$. Without loss of the generality, suppose $N = n_0$. By $\Delta^{i+1}x(n) \geq 0$, we have that $\Delta^i x(n)$ is monotonically nondecreasing in $[n_k, n_{k+1})$, $k = 0, 1, 2, \ldots$. Hence

$$\Delta^i x(n) \geq \Delta^i x(n_k), \quad n \in [n_k, n_{k+1}).$$

Summing the above inequality from n_k to $n_{k+1} - 1$, we have

$$\Delta^{i-1}x(n_{k+1}) \geq \Delta^{i-1}x(n_k) + \Delta^i x(n_k)(n_{k+1} - n_k). \tag{2.1}$$

So

$$\Delta^{i-1}x(n_2) \geq \Delta^{i-1}x(n_1) + \Delta^i x(n_1)(n_2 - n_1). \tag{2.2}$$

According to (2.2) and (B), we have

$$\begin{aligned}
\Delta^{i-1}x(n_3) &\geq \Delta^{i-1}x(n_2) + \Delta^i x(n_2)(n_3 - n_2) \\
&\geq \Delta^{i-1}x(n_1) + \Delta^i x(n_1)(n_2 - n_1) + g_{i,2}(\Delta^i x(n_2 - 1))(n_3 - n_2) \\
&\geq \Delta^{i-1}x(n_1) + \Delta^i x(n_1)(n_2 - n_1) + a_{i,2}\Delta^i x(n_2 - 1)(n_3 - n_2) \\
&\geq \Delta^{i-1}x(n_1) + \Delta^i x(n_1)(n_2 - n_1) + a_{i,2}\Delta^i x(n_1)(n_3 - n_2).
\end{aligned}$$

By induction, we get

$$\Delta^{i-1}x(n_k) \geq \Delta^{i-1}x(n_1) + \Delta^i x(n_1)\left[(n_2 - n_1) + a_{i,2}(n_3 - n_2) + \cdots + a_{i,2}a_{i,3} \cdots a_{i,k-1}(n_k - n_{k-1})\right].$$

From (H), we know that there exists l such that $\Delta^{i-1}x(n_k) > 0$ for $k \geq l$. Since $\Delta^i x(n) > 0$, we obtain

$$\Delta^{i-1}x(n) > \Delta^{i-1}x(n_k) > 0, \quad n \in [n_k, n_{k+1}),\ n_k \geq n_l.$$

If $\Delta^{i+1}x(n) \leq 0$, $\Delta^i x(n) < 0$ for $n \leq N$, then a similar reasoning of the previous proof implies the other conclusion. We omit the details to avoid repetition. □

Lemma 2.2. *Let $x(n)$ be a solution of Eq. (1.1) and (H) hold. Given $i \in \{1, 2, 3\}$, suppose there exists a constant $N (N \geq n_0)$ such that $x(n) > 0$, $\Delta^i x(n) \leq 0$, $\Delta^i x(n) \not\equiv 0$ in any interval $[n, \infty)$ for $n \geq N$. Then $\Delta^{i-1}x(n) > 0$ holds for sufficiently large n.*

Proof. Without loss of generality, we assume $N = n_0$. We first prove that $\Delta^{i-1}x(n_k) > 0$ holds for any $n_k \geq n_0$. Otherwise, we can choose $n_j > n_0$ such that $\Delta^{i-1}x(n_j) \leq 0$. From $\Delta^i x(n) \leq 0$, we get that $\Delta^{i-1}x(n)$ is nonincreasing in any $[n_k, n_{k+1})$.