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Vortex design problem

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ABSTRACT

In this investigation we propose a computational approach for the solution of optimal control problems for vortex systems with compactly supported vorticity. The problem is formulated as a PDE-constrained optimization in which the solutions are found using a gradient-based descent method. Recognizing such Euler flows as free-boundary problems, the proposed approach relies on shape differentiation combined with adjoint analysis to determine cost functional gradients. In explicit tracking of interfaces (vortex boundaries) this method offers an alternative to grid-based techniques, such as the level-set methods, and represents a natural optimization formulation for vortex problems computed using the contour dynamics technique. We develop and validate this approach using the design of 2D equilibrium Euler flows with finite-area vortices as a model problem. It is also discussed how the proposed methodology can be applied to Euler flows featuring other vorticity distributions, such as vortex sheets, and to time-dependent phenomena.

1. Introduction

There is a renewed interest in the computation of inviscid vortex flows featuring vorticity distributions more complicated than point vortices, namely, vortex sheets and vortex patches. Although still mostly limited to two-dimensional (2D) flows, these recent investigations are, on the one hand, motivated by emerging biomechanical applications where one-dimensional (1D) vortex sheets serve as models of the vortex wake generated by a swimming object; see, e.g., [1–6]. On the other hand, such studies are inspired by recent developments in computational complex analysis [7]. In addition, solutions of 2D Euler equations characterized by piecewise constant vorticity continue to find applications in the study of atmospheric and oceanographic phenomena [8,9]. From the mathematical point of view, a salient feature of all of these models is that they are described by partial differential equations (PDEs) of the free-boundary type in which the shape of the interface (i.e., the vortex sheet, or the boundary of the vortex patch) is a priori unknown and must be determined as a part of the solution of the problem. Computation of such systems is typically based on various versions of the "contour dynamics" approach [10] which has been significantly improved and generalized since its inception. At the same time, over the last decade or so there has been significant progress as regards the solution of a range of optimization and optimal control problems for fluid systems [11]. Most of the approaches proposed relied on the solution of suitably defined adjoint equations to determine the gradient of the cost functional to be minimized, and were usually focused on fixed-boundary problems. While there have been a number of investigations addressing optimization of the shape of the flow domain [12–16], we are not aware of any results concerning optimization of flow problems with internal interfaces, with the exception of Refs. [17,18] which however concern a rather different physical problem. Therefore, a long-term objective of the present research effort is to develop an optimization framework suitable for vortex dynamics problems of the type mentioned above. Since solving such optimization problems will typically involve constructing vortex systems with some prescribed properties, we will refer to this broad set of problems as "vortex design". It should be emphasized, however, that the techniques developed in

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the present study are applicable to the inviscid case only, as vorticity fields in viscous flows may not have discontinuities. Optimization problems for flows at finite Reynolds numbers are, at least in principle, amenable to solution using standard methods of adjoint-based optimization and we refer the reader to the monograph [11] for a survey and further references.

The problem of controlling and optimizing vortex configurations has already received some attention in the literature, and these efforts were surveyed in a recent review paper [19]. While these earlier investigations were concerned almost exclusively with systems of *point vortices*, here we seek to develop a systematic approach for the optimal control of vortices with more complicated vorticity distributions such as vortex sheets and vortex patches. More specifically, in the present investigation we introduce our approach on the basis of arguably the simplest problem in this class, namely, a steady-state flow with finite-area vortex patches (in fact, dealing with finite-length vortex sheets is technically more complicated due to the presence of the endpoints which act as geometric singularities, and is the subject of ongoing research). A key novelty of our approach is that, recognizing that such systems are in fact described mathematically by equations of the *free-boundary* type, our optimization methodology is developed based on methods of the "shape calculus". The shape calculus is a suite of techniques which allow one to treat PDE problems defined on variable domains and/or involving interfaces [20,21]. This appears as a natural way to frame an optimization problem for a vortex system, consistent with the "contour dynamics" approach typically employed for solving the "direct" problem of determining the time evolution or the steady states. In this sense, the proposed approach is an alternative to grid-based techniques such as those based on the level-set method [22]. In order to illustrate this new framework, in this paper we solve a design (inverse) problem for a vortex system in equilibrium with solid boundaries described by the 2D steady-state Euler equations. The structure of the paper is as follows: in the next section we introduce a class of steady-state solutions of 2D Euler equations known as the Prandtl–Batchelor flows which will be used as our model vortex system, in the following section we formulate the vortex design problem mathematically, in Section 4 we introduce elements of the shape calculus and establish the optimization framework, in Section 5 we discuss some numerical aspects of the solution of the optimization problem, whereas the computational results are presented in Section 6; discussion and conclusions are deferred to Sections 7 and 8, respectively.

2. Prandtl–Batchelor flow as a model vortex system

As is well known [23–25], the streamfunction ψ in the 2D steady-state Euler flows satisfies the following boundary value problem:

$$\Delta\psi = f(\psi) \quad \text{in } \Omega, \tag{1a}$$

$$\psi = \psi_b \quad \text{on } \partial\Omega, \tag{1b}$$

where $\Omega \subset \mathbb{R}^2$ is the flow domain, whereas $\psi_b : \partial\Omega \to \mathbb{R}$ is the boundary value of the streamfunction consistent with the prescribed boundary condition V_b^n for the wall-normal velocity component, i.e., $V_b^n \triangleq \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = \frac{\partial\psi}{\partial s}|_{\partial\Omega}$ in which $\mathbf{v} = [u, v] \triangleq [\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}]$, $\mathbf{n}$ is the unit vector normal to $\partial\Omega$ and pointing into the domain Ω, and s is the arc-length coordinate along $\partial\Omega$ (the symbol "$\triangleq$" means "equal by definition to"). The function $f : \mathbb{R} \to \mathbb{R}$ is not a priori prescribed and only has to meet some rather mild regularity conditions [24]. We note that its indeterminacy is a signature of the lack of uniqueness of solutions of the Euler equations. A common choice of the function f, motivated by the Prandtl–Batchelor hypothesis [26,27], is as follows:

$$f(\psi) = -\omega H(\psi_0 - \psi), \tag{2}$$

where $\omega, \psi_0 \in \mathbb{R}$ are two parameters and $H(\cdot)$ is the Heaviside function. We remark that with the form of $f(\psi)$ given in (2), the solutions of (1) feature regions of constant vorticity ω bounded by the streamline with $\psi = \psi_0$ and embedded in an otherwise irrotational (potential) flow (region A in Fig. 1). Evidently, solutions to (1)–(2) are characterized by two parameters, ω and ψ_0, or equivalently, the circulation of the vortex $\Gamma \triangleq \omega\int_\Omega H(\psi_0 - \psi)\, d\Omega$ and its area $|A| \triangleq \int_\Omega H(\psi_0 - \psi)\, d\Omega$. In addition to an analytical solution of (1)–(2) available in the form of the Rankine vortex [24], two-parameter families of solutions were found numerically, for example, in [28] for a counter-rotating vortex pair in an unbounded domain, and in [29] for the case of two counter-rotating vortices in equilibrium with a circular cylinder and a uniform flow at infinity. By fixing the circulation Γ of an individual vortex in these solutions, one obtains a family of flows desingularizing, respectively, a pair of point vortices and the Föppl system [30], which are recovered in the limit $|A| \to 0$ (or, equivalently, $\omega \to \pm\infty$). One such family of solutions of (1)–(2) desingularizing the Föppl system computed originally in [29] is shown in Fig. 2. Some questions concerning the conditions under which solutions of (1)–(2) can be continued with respect to their parameters were recently addressed in [31]. Hereafter we will only consider problems with zero net mass flux across the domain boundary $\partial\Omega$, so the boundary data ψ_b must satisfy the condition

$$\int_{\partial\Omega} \frac{\partial\psi_b}{\partial s}\, ds = \int_{\partial\Omega} \mathbf{v}\cdot\mathbf{n}\, ds = 0. \tag{3}$$

For some technical reasons (cf. [31]) we will assume that the vortex boundary ∂A is smooth; however, the boundary of the flow domain $\partial\Omega$ may have corners, although cusps are not allowed. There are also no restrictions on the connectivity of the flow domain Ω. As is evident from Fig. 2, Euler flows characterized by finite-area vortices have qualitatively quite different properties than the limiting point-vortex systems. We emphasize that the point-vortex systems have in fact the form