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Numerical method for solving diffusion-wave phenomena

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ABSTRACT

We find solutions for the diffusion-wave problem in $1D$ with n-term time fractional derivatives whose orders belong to the intervals $(0, 1)$, $(1, 2)$ and $(0, 2)$ respectively, using the method of the approximation of the convolution by Laguerre polynomials in the space of tempered distributions. This method transfers the diffusion-wave problem into the corresponding infinite system of linear algebraic equations through the coefficients, which are uniquely solvable under some relations between the coefficients with index zero.

The method is applicable for nonlinear problems too.

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1. Introduction

1.1. Preliminaries

Anomalous processes in time are introduced in [1,2]. Anomalous relaxation and diffusion phenomena anomalous due to time are the subject of [3]. Fractional diffusion equations which are described by anomalous diffusion processes have for the characteristic displacement scales which are a power of time. With a lack of it we have the diffusion equation with fractional derivatives of distributed order.

Fick's diffusion equation describes normal kinetic processes. It is a parabolic partial differential equation for the probability density function $u(x, t)$ and k is positive constant,

$$\partial_t u(x, t) = k^2 \partial_x^2 u(x, t).$$

In many cases which are called anomalous kinetics the characteristic displacement scale behaves as t^β, β is different from 1 or does not scale at all.

Proper scaling behavior described through the diffusion equations with fractional derivative (w.r.t.) temporal or spatial variables for anomalous diffusion are the Levy flights and the continuous time random walks with power law waiting time distributions (cf. [4,5]). Recall the equations concerning this.

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(1) Time fractional equations with Caputo fractional derivative (in notation D_*^β (w.r.t.) the temporal variable),

$$D_*^\beta u(x,t) = k^2 \partial_x^2 u(x,t), \quad (0 < \beta < 1), \tag{1}$$

k is a fractional diffusion constant $[k] = \mathrm{cm}^2/\mathrm{s}^\beta$;

(2) Space fractional diffusion equation with Riemann–Louville or Riesz derivative on the *RHS*

$$\partial_t u(x,t) = k^2 \partial_x^{2/\beta} u(x,t), \quad (\beta > 1). \tag{2}$$

These forms are known as normal. For modified forms of these equations cf. [3].

Nonscaling anomalous diffusion processes (cf. [6–8,2]) are referred to truncated Levy flights and Sinai superslow diffusion. The behavior of the corresponding probability density function is described by a diffusion equation with distributed order derivatives (cf. [6,7]). Distributed order derivative is a linear operator (cf. [2]) defined as a weight sum of different fractional derivatives or an integral of such over their order, $\int_a^b \mathrm{d}\beta p(\beta)\partial_z^\beta$ acting on the function of the corresponding variable z, where z means time or space.

In [3] modified distributed order equations are considered with both temporal and spatial fractional derivatives showing transformation of the anomalous solution at small times into the normal solution as a long time behavior. This has an application in biophysics, plasma physics and econophysics.

Mathematical foundation, existence–uniqueness and construction of the solution to the Cauchy problem for general linear evolution equation with temporal fractional derivatives of distributed orders are established in [9,10]. The following equations are related:

(i) time fractional equation with Laplace operator (w.r.t.) spatial variable of the form (1) for $0 < \beta < 1$ and $1 < \beta < 2$;

(ii) space fractional differential equations of the type (2) for $0 < \beta < 1$. In the one-dimensional case and under the condition $0 < \beta \leq 1$ this equation describes Levy–Feller diffusion processes, which is Markovian (cf. [8]). Feller semigroups are constructed (cf. [11,12]), for more general operators on the *RHS*.

(iii) time and space fractional equation

$$D_*^\beta u(x,t) = D_0^\alpha u(x,t), \quad x \in \mathbf{R}^n, \ \alpha, \beta > 0,$$

describing anomalous diffusion processes which is non-Markovian in character (cf. [13]), and modified equations in [3]:

$$\partial_t u(x,t) = k^2 D^{1-\beta} \partial_x^2 u(x,t);$$

where $0 < \beta < 1$ for subdiffusion and for superdiffusion

$$D^{1-1/\beta} \partial_t u(x,t) = k^2 \partial_x^2 u(x,t);$$

when $\beta > 0$ with the spatial Riemann–Louville derivative or Riesz derivative for superdiffusion.

Recently, anomalous diffusion is studied (cf. [14]) by fractal derivatives giving the fundamental solutions which show clear power law characteristics. It is given fractal derivative modelling of anomalous diffusion

$$D_t^\beta u(x,t) = \mathrm{d}\partial_x^\alpha u(x,t), \quad t > 0, \ 0 < \beta, \ \alpha \leq 2, \qquad u(x,0) = \delta(x), \quad -\infty < x < \infty.$$

Fractal and fractional derivatives are two important approaches in modelling of anomalous diffusion. The fractal derivative has local properties like entire derivatives and it is simpler and computationally more efficient than the fractional derivative which has global properties. The fractal model is slower at the initial period but faster in the long-term evolution than the fractional model and possesses different symmetry of the diffusion.

Finally, in [9], the diffusion-wave phenomena are described with applications in physics, related to the sub-diffusion with retardation studied in [2],

$$b_1 D_*^{\beta_1} u(x,t) + b_2 D_*^{\beta_2} u(x,t) = k^2 \partial_x^2 u(x,t), \quad 0 < \beta_1 < \beta_2 \leq 1, \ b_1, b_2 > 0, \ b_1 + b_2 = 1, \tag{3}$$

and its generalization to n-time fractional equations obtained by setting finite sum of delta distributions as weight function $p(\beta) = \sum_{i=0}^m b_i \delta(\beta - \beta_i)$, where $i < \beta_i \leq i+1, \ i = 0, 1, \ldots, m-1$, into the diffusion-wave problem of the distributed order (5)–(6) i.e. the equation

$$b_0 D_*^{\beta_0} u(x,t) + \sum_{i=1}^{m-1} b_i D_*^{\beta_i} u(x,t) + b_m D_*^{\beta_m} u(x,t) = k^2 \partial_x^2 u(x,t), \quad t \in (0,T), \tag{4}$$

where $x \in \mathbf{R}^n, \ b_0, \ldots, b_m \in \mathbf{R}_+$, which is the generalization to the problem (3).

In this paper we give a numerical method for solving Eqs. (3) respectively (4), based on orthogonal polynomials of the Laguerre type. It is only the description of how the method works. Note that all equations from this Section 1.1 can be solved in 1*D* by the method of the approximation of the tempered convolution. Many temporal–spatial fractional equations with fractional and entire derivatives have a solution by this method if the spatial variable has a dimension one. The method is applicable for the nonlinear equations of this type and the nonlinear equations with the forcing term of the corresponding growth. The application to nonlinear problems whose nonlinear term is of the Lipschitz's class is also possible. Nonlinear time fractional diffusion-wave equations obtained by perturbing the equation with nonlinear terms, singular distribution, or stochastic processes (say, fractional derivative of the Brownian motion) also have a solution as an application of this method.

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