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Multi-stage solvers optimized for damping and propagation

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ABSTRACT

Explicit multi-stage solvers are routinely used to solve the semi-discretized equations that arise in Computational Fluid Dynamics (CFD) problems. Often they are used in combination with multi-grid methods. In that case, the role of the multi-stage solver is to efficiently reduce the high frequency modes on the current grid and is called a smoother. In the past, when optimizing the coefficients of the scheme, only the damping characteristics of the smoother were taken into account and the interaction with the remainder of the multi-grid cycle was neglected. Recently it had been found that coefficients that result in less damping, but allow for a higher Courant–Friedrichs–Lewy (CFL) number are often superior to schemes that try to optimize damping alone. While this is certainly true for multi-stage schemes used as a stand-alone solver, we investigate in this paper if using higher CFL numbers also yields better results in a multi-grid setting. We compare the results with a previous study we conducted and where a more accurate model of the multi-grid cycle was used to optimize the various parameters of the solver.

We show that the use of the more accurate model results in better coefficients and that in a multi-grid setting propagation is of little importance.

We also look into the gains to be made when we allow the parameters to be different for the pre- and post-smoother and show that even better coefficients can be found in this way.

1. Introduction

Explicit multi-stage schemes are commonly used to solve ordinary differential equations resulting from the space discretization of partial differential equations. When used as a stand-alone solver, such a scheme can either be designed for high order accuracy or to reach steady state quickly when the order of accuracy is less important. As it can easily be shown that low frequency error components are difficult to damp, a multi-stage scheme that improves the propagation of these components generally yields a faster solver; a characteristic of these solvers is the higher CFL number.

When a multi-stage iteration is used in conjunction with multi-grid, the aim is to reduce those low frequency components on a coarser grid; if the number of nodes on the coarser grid is low, a direct solution on that grid can be cost-effective.

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The multi-stage solver is then often designed to eliminate the high frequency components of the error (which cannot be represented on the coarser grid) and is called a smoother. In the past, these coefficients have been optimized by trial and error [1] or by geometric methods [2].

These and other studies [3–5] focused on minimizing the smoothing factor over the high frequency domain. Still, it was realized that designing the multi-stage scheme to be an optimal smoother, as would be desirable for multi-grid to work, would not result in an optimal (i.e. fastest) overall scheme, but that some precedence had to be given to the propagative behavior [6]. Recently the optimization of the damping and propagative efficiencies was expressed within the framework of constrained non-linear optimization [7]. Different objective functions with a variety of constraints were constructed on a heuristical basis and their effect on the convergence rate of the numerical scheme was studied. Multi-stage coefficients that are comparable with values used in the past were found.

In [8] we chose to look at a complete 2-grid cycle which closely modeled the interaction between multi-stage solver, defect correction, restriction and prolongation. Multi-stage solvers with an improved performance were found, although their smoothing capacity was sacrificed to some extent.

In this paper we refine the model established in [8] by also looking at the propagation characteristics. Again we are only interested in reaching convergence quickly using a multi-grid scheme, and therefore focus on the multi-stage scheme for which the 2-grid cycle will remove all frequencies as quickly as possible, either by propagation or by damping. We focus on the 1 dimensional advection equation; this limitation is justified, as Hosseini and Alonso [7] have shown that 1D wave equations can serve as a good indication of the performance of real flow solvers.

This paper is organized as follows. In Section 2 we formulate the equation under consideration, the Fourier footprints of the space discretizations and the transmittance function of the multi-stage time-stepping scheme; in Section 3 we model the remainder of the 2-grid cycle and give the transmittance function of its components; in Section 4 we deduce the propagation and damping that results from a transmittance function and formulate the analytical framework for optimization; in Section 5 the results are analyzed in two cases: identical or different parameters for pre- and post-smoother.

Remark. We use $\mathbf{i}$ as a symbol for the complex unit ($=\sqrt{-1}$) and i as an index.

2. The model: the scalar 1D advection equation

2.1. General formulation

We are interested in the advection equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1}$$

to which we add a suitable boundary condition, which in our case are the inlet value of u. Applying a given space discretization to (1) on an equidistant mesh with N nodes results in the semi-discretized nodal equation ($j = 1, \dots, N$):

$$\frac{\mathrm{d}\tilde{u}_j}{\mathrm{d}t} + \check{r}_j = 0 \tag{2}$$

where the space discretization operator $\check{r}_j$, which is a residual for the steady state solution, is of the form

$$\check{r}_j = \sum_{k=k^-}^{k^+} \beta_{j+k}\tilde{u}_{j+k} \tag{3}$$

where k^- and k^+ depend on the stencil that is used for the space discretization.

If $u_{\mathrm{exact},j}$, is the solution to (2), we can write $u_j = u_{\mathrm{exact},j} + e_j$, where e_j is the (nodal) error of u_j with respect to $u_{\mathrm{exact},j}$. Eq. (2) can then be transformed into the form

$$\frac{\mathrm{d}\tilde{e}_j}{\mathrm{d}t} + \underbrace{\sum_{k=k^-}^{k^+} \beta_{j+k}\tilde{e}_{j+k}}_{r_j} = 0. \tag{4}$$

We assume that $[\tilde{e}_1 \quad \tilde{e}_2 \quad \cdots \quad \tilde{e}_N]$ can be written as an infinite sum of Fourier modes, of which we now only consider the p-th: $\tilde{e}_{j,p} = \hat{e}_p(t)\mathrm{e}^{\mathbf{i}jp\Delta x}$ ($j = 1, \dots, N$). A similar expression will be used for r_j. Alternatively, when introducing the phase angle $\theta = p\Delta x$ we get $\tilde{e}_{j,\theta} = \hat{e}_\theta(t)\mathrm{e}^{\mathbf{i}j\theta}$ with $\theta \in [-\pi, \pi]$ ($j = 1, \dots, N$).

Inserting this in Eq. (4) we get an expression for the amplitude of the error $\hat{e}_\theta$ of the form

$$\frac{\mathrm{d}\hat{e}_\theta}{\mathrm{d}t} = -\hat{r}_\theta = \lambda(\theta)\hat{e}_\theta \tag{5}$$