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Quenching for nonlinear degenerate parabolic problems

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ABSTRACT

For the problem given by $u_\tau = \left(\xi^r u^m u_\xi\right)_\xi / \xi^r + f(u)$ for $0 < \xi < a$, $0 < \tau < \Lambda \le \infty$, $u(\xi, 0) = u_0(\xi)$ for $0 \le \xi \le a$, and $u(0, \tau) = 0 = u(a, \tau)$ for $0 < \tau < \Lambda$, where a and m are positive constants, r is a constant less than 1, $f(u)$ is a positive function such that $\lim_{u \to c^-} f(u) = \infty$ for some positive constant c, and $u_0(\xi)$ is a given function satisfying $u_0(0) = 0 = u_0(a)$, this paper studies quenching of the solution u.

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1. Introduction

Let a and m be positive constants, r be a constant less than 1, and Λ be a number such that $0 < \Lambda \le \infty$. In this paper, we study the existence and uniqueness of the solution and quenching phenomenon for the following nonlinear degenerate parabolic problem with the initial-Dirichlet boundary-value condition,

$$u_\tau = \frac{1}{\xi^r}\left(\xi^r u^m u_\xi\right)_\xi + f(u) \quad \text{in } (0, a) \times (0, \Lambda), \tag{1.1}$$

$$u(\xi, 0) = u_0(\xi) \quad \text{on } [0, a], \qquad u(0, \tau) = 0 = u(a, \tau) \quad \text{for } \tau \in (0, \Lambda), \tag{1.2}$$

where $f(u)$ and $u_0(\xi)$ are given functions. Since the coefficient of $u_{\xi\xi}$ is u^m, which tends to zero when ξ approaches 0 and a. We can regard (1.1) as a degenerate equation. Let $\xi = ax$, $\tau = a^2(m+1)t$, $\Lambda = a^2(m+1)T$, $D = (0, 1)$, $\Omega = D \times (0, T)$, $\bar{D} = [0, 1]$, $\bar{\Omega} = \bar{D} \times [0, T)$, and $\partial\Omega = \left(\bar{D} \times \{0\}\right) \cup (\{0, 1\} \times (0, T))$. Then, the problem (1.1)–(1.2) is formulated below,

$$u_t = \left(u^{m+1}\right)_{xx} + \frac{r}{x}\left(u^{m+1}\right)_x + a^2(m+1)f(u) \quad \text{in } \Omega, \tag{1.3}$$

$$u(x, 0) = u_0(x) \quad \text{on } \bar{D}, \qquad u(0, t) = 0 = u(1, t) \quad \text{for } t \in (0, T). \tag{1.4}$$

We assume that $u_0(x)$ is a positive function in D such that $u_0(0) = 0 = u_0(1)$, $u_0^{m+1}(x) \in C^{2+\alpha}\left(\bar{D}\right)$ for some $\alpha \in (0, 1)$, $\max_{x \in \bar{D}} u_0(x) < c$ for some positive constant c, and

$$\left(u_0^{m+1}\right)'' + \frac{r}{x}\left(u_0^{m+1}\right)' + a^2(m+1)f(u_0) \ge 0 \quad \text{in } D. \tag{1.5}$$

We also suppose that the source term $f \in C^2([0, c))$, $f(0) > 0$, $f'(0) > 0$, $f''(s) > 0$ for $s \in [0, c)$, and $\lim_{u \to c^-} f(u) = \infty$.

The study of the problem (1.1)–(1.2) is motivated by the paper of Gratton et al. [1]. When $f(u) = 0$ and $r \ge 0$, they used (1.1) to describe the creeping gravity flow of a power-law liquid on a rigid horizontal surface. In their paper, they considered planar and axisymmetric flows. u was the thickness of the current, and r represented the Cartesian symmetry. When u is

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considered as the temperature of an object, (1.1) can be interpreted as a nonlinear heat conduction, and u^m represents the thermal diffusivity (cf. [2, pp. 73–74]).

When $f(u) = u^p$, Chan and Chan [3] studied the existence of classical solutions of the problem (1.1)–(1.2). When $m = 0$, Chan and Chen [4] studied the quenching of the problem (1.1)–(1.2) for $f(u) = 1/(1-u)$. They used a lower bound of u to estimate the critical length of u numerically. They also determined an upper bound of the quenching time.

In Section 2, we shall discuss the existence and uniqueness of the classical solution of the problem (1.3)–(1.4). We shall show the comparison theorem for the problem (1.3)–(1.4). In Section 3, we shall prove that u will quench in a finite time under some conditions.

2. Existence and uniqueness of the solution

To study the existence of the classical solution of the problem (1.3)–(1.4), we let $p = u^{m+1}$. Then, p satisfies the following nonlinear degenerate parabolic problem,

$$p_t = (m+1)\, p^{m/(m+1)} \left[p_{xx} + \frac{r}{x} p_x + a^2 (m+1) f(p) \right] \quad \text{in } \Omega, \tag{2.1}$$

$$p(x,0) = p_0(x) \quad \text{on } \bar{D}, \qquad p(0,t) = 0 = p(1,t) \quad \text{for } t \in (0,T), \tag{2.2}$$

where $p_0(x) = u_0^{m+1}(x)$. Since the coefficient $(m+1)\, p^{m/(m+1)}$ tends to 0 when x approaches 0 or 1, (2.1) is not uniformly parabolic. To establish the existence of the classical solution of the problem (2.1)–(2.2), let ε be a small positive number less than 1, and p_ε satisfy the following nonlinear problem,

$$p_{\varepsilon_t} = (m+1)(p_\varepsilon + \varepsilon)^{m/(m+1)} \left[p_{\varepsilon_{xx}} + \frac{r}{x} p_{\varepsilon_x} + a^2 (m+1) f(p_\varepsilon) \right] \quad \text{in } \Omega, \tag{2.3}$$

$$p_\varepsilon(x,0) = p_0(x) \quad \text{on } \bar{D}, \qquad p_\varepsilon(0,t) = 0 = p_\varepsilon(1,t) \quad \text{for } t \in (0,T). \tag{2.4}$$

To prove the existence of the solution of the problem (2.3)–(2.4), let us construct a sequence $\{w_i\}$ as follows: $w_0(x) = p_0(x)$, and w_i satisfies the following linear parabolic initial-Dirichlet boundary-value problem for $i = 1, 2, 3, \ldots$

$$w_{i_t} = (m+1)(w_{i-1} + \varepsilon)^{m/(m+1)} \left[w_{i_{xx}} + \frac{r}{x} w_{i_x} + a^2 (m+1) f(w_{i-1}) \right] \quad \text{in } \Omega, \tag{2.5}$$

$$w_i(x,0) = p_0(x) \quad \text{on } \bar{D}, \qquad w_i(0,t) = 0 = w_i(1,t) \quad \text{for } t \in (0,T). \tag{2.6}$$

When $i \to \infty$, we want to prove that $\{w_i\}$ converges to the unique classical solution p_ε. In addition, when $\varepsilon \to 0$, we show that $\{p_\varepsilon\}$ converges to a classical solution p. With this result, we then establish the existence of a classical solution u. From (1.5), w_0 satisfies

$$w_0'' + \frac{r}{x} w_0' + a^2 (m+1) f(w_0) \ge 0 \quad \text{in } D. \tag{2.7}$$

Clearly, w_0 is a lower solution of the problem (2.5)–(2.6). We use the steady state solution $v(x)\ (< c)$ being the upper solution of the problem (2.5)–(2.6) where $v(x)$ satisfies the following two-point boundary-value problem,

$$\frac{1}{x^r} \frac{d}{dx} \left(x^r \frac{dv}{dx} \right) = -a^2 (m+1) f(v) \quad \text{in } D, \quad v(0) = 0 = v(1). \tag{2.8}$$

Existence of v is obtained by the monotone sequence $\{v_i\}_{i=0}^{\infty}$ (cf. [4]) which satisfies: $v_0 \equiv 0$ and for $i = 1, 2, 3, \ldots$

$$\frac{d}{dx} \left(x^r \frac{dv_i}{dx} \right) + \sigma x^r v_i = \sigma x^r v_{i-1} - a^2 (m+1) x^r f(v_{i-1}) \quad \text{in } D, \quad v_i(0) = 0 = v_i(1),$$

where σ is a positive constant. By modifying the proof of Theorem 5 of Chan and Chen and Lemma 3 of [5], we obtain the following lemma.

Lemma 2.1. *If $\sigma \le 1$ and $a^2 (m+1) f'(0) \ge \sigma$, then the two-point boundary-value problem (2.8) has the unique solution $v \in C(\bar{D}) \cap C^2((0,1])$ and $0 < v < c$ in D, to which the sequence $\{v_i\}_{i=0}^{\infty}$ converges to v.*

Let $k_1, k_2, \ldots, k_{17}$ denote appropriate positive constants. In what follows, we assume that

$$w_0(x) \le v(x) < c \quad \text{for } x \in \bar{D}.$$

Let $E = D \times [0,T)$. We obtain the following properties for w_i.

Lemma 2.2. *For any positive integer i,*

(i) $v(x) \ge w_i(x,t) \ge w_0(x)$ *on* $\bar{\Omega}$,
(ii) $w_i \in C(\bar{\Omega}) \cap C^{2+\alpha, 1+\alpha/2}(E)$ *and is unique,*
(iii) $w_{i_t} \ge 0$ *on* $\bar{\Omega}$,
(iv) $\{w_{i-1}\}$ *is a monotone nondecreasing sequence on* $\bar{\Omega}$.