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Evaluating American put options on zero-coupon bonds by a penalty method

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ABSTRACT

In this paper, American put options on zero-coupon bonds are priced under a single factor model of short-term rate. The linear complementarity problem of the option value is solved numerically by a penalty method, by which the problem is transformed into a nonlinear PDE by adding a power penalty term. The solution of the penalized problem converges to that of the original problem. A numerical scheme is established by using the finite volume method and the corresponding stability and convergence are discussed. Numerical results are presented to show the usefulness of the method.

1. Introduction

It is known that many term structure models for the short-term interest rate $r(t)$ can be nested within the CKLS model [1] as defined by

$$\mathrm{d}r(t) = \kappa(\alpha - r(t))\mathrm{d}t + \sigma r(t)^{\gamma}\mathrm{d}W(t) \tag{1}$$

where $W(t)$ is a Wiener process, κ is the speed of mean-reversion, α is the long-term interest rate, and σ is the volatility. Generally speaking, κ, α, σ could be constants or functions of t. For simplicity, they are assumed to be constants and $\kappa\alpha \neq 0$ here. In the case $\gamma = 0$ or 0.5, model (1) is the Vasicek model or the CIR model, respectively, under which analytic expressions for zero-coupon bond price exist. The short-term rate r can be negative in the Vasicek model, while it is non-negative in the CIR model. Both models are pioneering work on the modeling of short-term rate. Chan [1] claimed that the CKLS model with $\gamma \geq 1$ captures the dynamics of the short-term interest rate better than the model with $\gamma < 1$ when fitting monthly US treasury bill yield from June 1964 to December 1989. Nowman and Sorwar [2] found that when fitting Canada, Hong Kong, and United States currency rates during February 1981 and December 1997, γ varies from 0.0076 to 1.2260. In particular, γ is 1.1122 and 1.2660, respectively, for 1- and 3-month United States rates.

In the case $\gamma = 0.5$, zero-coupon bonds can be priced analytically, and the pricing of American put options have been studied in [3], where both finite volume method and finite element method are used for setting up numerical schemes and whose stability and convergence are analyzed. Brennan–Schwartz algorithm [4] is used for evaluating the option value. In the case $\gamma = 1$, there is no analytic solution for the zero-coupon bond price, which will be used in initial and boundary conditions when valuing options on the bond. It needs to be computed numerically. American put options on zero-coupon bonds under the assumption $\gamma = 1$ are tackled in [5], also with the Brennan–Schwartz algorithm for the option value. When $\gamma \geq 1$, European call options on zero-coupon bonds are investigated in [6]. To the best of our knowledge, American put options on bonds are all priced by projected successive overrelaxation method (PSOR) [7] or the Brennan–Schwartz algorithm in the literature. In this paper, a novel penalty method for pricing American options on zero-coupon bonds is

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proposed, by which the linear complementarity problem of the option value is transformed into a nonlinear PDE by adding a power penalty term. The convergence of the penalized solution to the real solution is guaranteed by the results of [8]. A numerical scheme for the penalized problem is established and numerical examples are provided.

In our study, the parameter γ varies from 0.3912 to 1.2660, which is produced by fitting the CKLS model to the actual market rates [2], and we use the power penalty approach [8] to solve the linear complementarity problem for the option value. The main advantages of the penalty approach are: (i) when sophisticated discretization methods other than standard central or upstream weighting methods are employed, it would reduce computational cost and time; (ii) it may converge faster than the PSOR method for American options with early exercise constraint. In view of the fact that no explicit closed-form solution exists for the American put option, we examine the accuracy of our method by comparing it with other numerical techniques. We take the results computed by the Brennan–Schwartz algorithm as benchmark and find that our penalty approach provides both option value and optimal exercise interest rate with similar accuracy.

The rest of this paper is organized as follows. In Section 2, we introduce the pricing models of zero-coupon bond and American put option on the bond under the short rate model (1). In Section 3, we establish a numerical scheme by using the finite volume method and give the corresponding stability and convergence property. In Section 4, we illustrate the power penalty approach for the discretized linear complementarity problem and its convergence. In Section 5, the Brennan–Schwartz algorithm is described briefly. We give our numerical results in Section 6 and concluding remarks in Section 7.

2. The pricing model of zero-coupon bond and option

Under the interest rate process (1), we can derive the pricing equations for zero-coupon bonds and European options on the bonds using the traditional no-arbitrage argument. Actually, the pricing equations for both financial products are the same except for the boundary conditions [6]. We denote the price at time t of the zero-coupon bond with face value E and maturity T^* by $B(r,t,T^*)$, then it satisfies the following equation:

$$\frac{\partial B}{\partial t}+\frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 B}{\partial r^2}+\kappa(\alpha-r)\frac{\partial B}{\partial r}-rB=0, \tag{2}$$

$$B(r,T^*,T^*)=E. \tag{3}$$

Furthermore, let $V(r,t)$ represent the value of European put option on the above bond with maturity T and exercise price K, then it satisfies:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 r^{2\gamma}\frac{\partial^2 V}{\partial r^2}+\kappa(\alpha-r)\frac{\partial V}{\partial r}-rV=0, \tag{4}$$

$$V(r,T)=\max(K-B(r,T,T^*),0). \tag{5}$$

We consider an American put option on the zero-coupon bond with exercise price K and expiry date T $(<T^*)$. Although the underlying asset is the bond, the independent variable is the stochastic interest rate. The bond price is only used in the initial and boundary conditions. Similar to American put options on stocks, there is an unknown optimal exercise interest rate $r^*(t)$ at time t for the American put on a bond. It is the smallest value of the interest rate at which the exercise of the put option becomes optimal. The American put option value $P(r,t)$ satisfies the following free boundary problem [9]:

$$P_t+LP=0,\qquad P(r,t)>g(r,t),\quad 0<r<r^*(t),\ 0\le t<T, \tag{6}$$

$$P(r^*(t),t)=g(r^*(t),t),\qquad P_r(r^*(t),t)=g_r(r^*(t),t),\quad 0\le t<T, \tag{7}$$

$$P(r,t)=g(r,t),\quad r>r^*(t),\ 0\le t\le T, \tag{8}$$

$$P(0,t)=g(0,t),\quad 0\le t\le T, \tag{9}$$

$$P(r,T)=g(r,T),\quad r\ge 0, \tag{10}$$

where

$$g(r,t)=\max(K-B(r,t,T^*),0),$$

$$LP=\frac{1}{2}\sigma^2 r^{2\gamma}P_{rr}+\kappa(\alpha-r)P_r-rP.$$

$P_t=\frac{\partial P}{\partial t}$, $P_r=\frac{\partial P}{\partial r}$, $P_{rr}=\frac{\partial^2 P}{\partial r^2}$, $g_r=\frac{\partial g}{\partial r}$. K should be strictly less than $B(0,T,T^*)$, otherwise, exercise the option would never be optimal.

In order to solve the system defined by (6)–(10), we need to solve (2)–(3) for the bond price first by numerical method since no analytical solution can be derived. It is known that r is the short-term interest rate, we can restrict $r\in[0,R]$ for some sufficiently large number R, and define boundary conditions for (2) at $r=0$ and $r=R$ as the bond price determined by the CIR model (which has an analytic expression). The numerical scheme for (2) will be clarified in the next section.