Journal of Computational and Applied Mathematics 235 (2011) 4350–4367

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The Trefftz method using fundamental solutions for biharmonic equations

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ARTICLE INFO

Article history:
Received 18 September 2009
Received in revised form 23 February 2011

Keywords:
Method of fundamental solutions
Almansi's fundamental solutions
Biharmonic equation
Singularity problem
Trefftz method
Error and stability analysis

ABSTRACT

In this paper, the Trefftz method of fundamental solution (FS), called the method of fundamental solution (MFS), is used for biharmonic equations. The bounds of errors are derived for the MFS with Almansi's fundamental solutions (denoted as the MAFS) in bounded simply connected domains. The exponential and polynomial convergence rates are obtained from highly and finitely smooth solutions, respectively. The stability analysis of the MAFS is also made for circular domains. Numerical experiments are carried out for both smooth and singularity problems. The numerical results coincide with the theoretical analysis made. When the particular solutions satisfying the biharmonic equation can be found, the method of particular solutions (MPS) is always superior to the MFS and the MAFS, based on numerical examples. However, if such singular particular solutions near the singular points do not exist, the local refinement of collocation nodes and the greedy adaptive techniques can be used for seeking better source points. Based on the computed results, the MFS using the greedy adaptive techniques may provide more accurate solutions for singularity problems. Moreover, the numerical solutions by the MAFS with Almansi's FS are slightly better in accuracy and stability than those by the traditional MFS. Hence, the MAFS with the AFS is recommended for biharmonic equations due to its simplicity.

1. Description of MFS

For simplicity, first consider the homogeneous biharmonic equation with the clamped boundary conditions

$$\Delta^2 u = 0 \quad \text{in } S, \tag{1.1}$$
$$u = f \quad \text{on } \Gamma, \tag{1.2}$$
$$u_\nu = g \quad \text{on } \Gamma, \tag{1.3}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, S is the bounded simply connected domain, $u_\nu = \frac{\partial u}{\partial \nu}$ is the outward normal derivative to Γ, Γ is its boundary, and f and g are the known functions smooth enough. In real application, we may encounter the non-homogeneous equation $\Delta^2 u = p(x, y)$ in S. Suppose that a particular solution $\bar{u}$ is found so that $\Delta^2 \bar{u} = p(x, y)$ in S. By means of a transformation $w = u - \bar{u}$ we have the homogeneous biharmonic equation $\Delta^2 w = 0$ in S with the clamped

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doi:10.1016/j.cam.2011.03.024

boundary conditions $w = \bar{f} = f - \bar{u}$ on Γ and $w_\nu = \bar{g} = g - \bar{u}_\nu$ on Γ. Hence we may simply consider (1.1)–(1.3). The general solutions of biharmonic equations can be represented by

$$u = u(\rho, \theta) = \rho^2 v + z, \tag{1.4}$$

where (ρ, θ) are the polar coordinates, and v and z are the harmonic functions. Denote $r = |\overline{PQ}|$, $P = \rho e^{i\theta}$, $Q = Re^{i\varphi}$, φ is a radian with $0 \le \varphi \le 2\pi$, and $i = \sqrt{-1}$. Then $r = \sqrt{R^2 + \rho^2 - 2R\rho\cos(\theta - \varphi)}$. Hence the fundamental solutions of biharmonic equations in 2D are found from (1.4) as

$$\Phi(\rho, \theta) = r^2 \ln r = (R^2 + \rho^2 - 2R\rho\cos(\theta - \varphi)) \ln \sqrt{R^2 + \rho^2 - 2R\rho\cos(\theta - \varphi)}. \tag{1.5}$$

Denote

$$\Phi_j(\rho, \theta) = r_j^2 \ln r_j, \tag{1.6}$$

$$\phi_j(\rho, \theta) = \ln r_j, \tag{1.7}$$

where $r_j = |\overline{PQ_j}| = \sqrt{R^2 + \rho^2 - 2R\rho\cos(\theta - \varphi_j)}$ and $Q_j = Re^{i\varphi_j}$ with $\varphi_j \in [0, 2\pi]$. Hence we may choose the linear combinations of (1.6) and (1.7):

$$v_N = \sum_{j=1}^{N} \left\{ c_j \Phi_j(\rho, \theta) + d_j \phi_j(\rho, \theta) \right\}, \tag{1.8}$$

where c_j and d_j are the unknown coefficients to be determined by the boundary conditions (1.2) and (1.3). We may use the Trefftz method [1]. Denote V_N the set of (1.8). Then the Trefftz solution u_N is obtained by

$$I(u_N) = \min_{v \in V_N} I(v), \tag{1.9}$$

where the energy

$$I(v) = \int_\Gamma (v - f)^2 + w^2 \int_\Gamma (v_\nu - g)^2, \tag{1.10}$$

ν is the normal of Γ, and w is the weight chosen as $w = 1/N$ in computation.

Almansi's fundamental solutions (simply denoted Almansi's FS) for biharmonic equations are obtained directly from (1.4).

Then Almansi's FS is given by

$$\Phi^A(\rho, \theta) = \rho^2 \ln \sqrt{R^2 + \rho^2 - 2R\rho\cos(\theta - \varphi)}, \tag{1.11}$$

while the fundamental solutions (1.5) are called the traditional FS in this paper. We may choose the linear combination of (1.11) and (1.7):

$$v_N^A = \sum_{j=1}^{N} \left\{ c_j \Phi_j^A(\rho, \theta) + d_j \phi_j(\rho, \theta) \right\}, \tag{1.12}$$

to replace (1.8), where

$$\begin{aligned} \Phi_j^A(\rho, \theta) &= \rho^2 \ln r_j \\ &= \rho^2 \ln \sqrt{R^2 + \rho^2 - 2R\rho\cos(\theta - \varphi_j)}. \end{aligned} \tag{1.13}$$

The coefficients c_j and d_j can also be obtained from the Trefftz method (1.9). For the biharmonic equation, the MFS and numerical experiments are carried out for (1.8) and (1.12) in [2–5]. The other kind of fundamental solution is also introduced in [2].

Next, let us consider the mixed type of the clamped and simply support boundary conditions on Γ. Then Eq. (1.3) is replaced by (see [6])

$$u_\nu = g \quad \text{on } \Gamma_1, \qquad u_{\nu\nu} = g^* \quad \text{on } \Gamma_2, \tag{1.14}$$

where $\Gamma_1 \cup \Gamma_2 = \Gamma$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$. The admissible functions (1.8) and (1.12) remain, but the energy $I(v)$ in (1.9) is replaced by

$$I^*(v) = \int_\Gamma (v - f)^2 + w^2 \int_{\Gamma_1} (v_\nu - g)^2 + (w_1^*)^2 \int_{\Gamma_2} (v_{\nu\nu} - g^*)^2, \tag{1.15}$$

where the weight $w_1^* = w^2 = 1/N^2$.