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An inverse problem of identifying the coefficient of first-order in a degenerate parabolic equation[☆]

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ABSTRACT

This work studies an inverse problem of determining the first-order coefficient of degenerate parabolic equations using the measurement data specified at a fixed internal point. Being different from other ordinary parameter identification problems in parabolic equations, in our mathematical model there exists degeneracy on the lateral boundaries of the domain, which may cause the corresponding boundary conditions to go missing. By the contraction mapping principle, the uniqueness of the solution for the inverse problem is proved. A numerical algorithm on the basis of the predictor–corrector method is designed to obtain the numerical solution and some typical numerical experiments are also performed in the paper. The numerical results show that the proposed method is stable and the unknown function is recovered very well. The results obtained in the paper are interesting and useful, and can be extended to other more general inverse coefficient problems of degenerate **PDE**s.

1. Introduction

In this paper, we study an inverse problem of recovering the first-order coefficient of degenerate parabolic equations when the extra condition specified at a fixed internal point is given. Problems of this type have important applications in several fields of applied science and engineering. The problem can be stated in the following form:

Problem P. Consider the following parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - k^2(x)\dfrac{\partial^2 u}{\partial x^2} + \rho(t)k(x)\dfrac{\partial u}{\partial x} = 0, & (x,t) \in Q = [0,l] \times (0,T], \\ u|_{t=0} = \phi(x), & x \in [0,l], \end{cases} \tag{1.1}$$

where k and ϕ are two given smooth functions which satisfy

$$k(0) = k(l) = 0, \qquad k(x) > 0,\ x \in (0,l), \tag{1.2}$$

and

$$\phi(x), \phi'(x) \geq 0, \qquad \phi(x) \not\equiv 0, \quad x \in [0,l], \tag{1.3}$$

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and $\rho(t)$ is an unknown coefficient in (1.1). Assume that an additional condition is given as follows:

$$u(x_0, t) = g(t), \quad t \in [0, T], \tag{1.4}$$

where $g(t)$ is a known function which satisfies the compatibility condition

$$g(0) = \phi(x_0),$$

and $x_0 \in (0, l)$ is a fixed point. We shall determine the functions u and ρ satisfying (1.1) and (1.4).

Partial differential equations (**PDE**s) may be central to mathematics, whether pure or applied. They are widely applied in fluid dynamics, geophysics, heat transfer, electromagnetism, finance, quantum mechanics and other areas of applications. Most of these phenomena can be described by classical **PDE**s which are based on the basic assumption that the principle coefficients of equations are strictly positive. However, there also exist many other problems arising in engineering or finance, which may lead to degenerate **PDE**s. For example, the well-known Black–Scholes equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S)S^2\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \quad (S, t) \in R^+ \times [0, T) \tag{1.5}$$

is a basic tool in the option pricing (see [1,2]), where V is the option premium, S is the value of underlying assets, r is the risk-free rate of interest, q is the stock dividend yield and σ is the volatility function. It can be easily seen that Eq. (1.5) degenerates into an ordinary differential equation on the left boundary $S = 0$.

The main difference between the classical **PDE**s and the degenerate ones is that boundary conditions may be missing for degenerate **PDE**s, which is also the main difficulty of degenerate inverse problems. In other words, if a degenerate **PDE** needs no boundary conditions, then for some given boundary conditions the equation may have no solution (**overdetermined**). Likewise, if boundary conditions are necessary, then the equation without any restriction may have many solutions (**undetermined**). In fact, by the well known Fichera's theory (see [3]), we know that whether or not boundary conditions should be given is determined by the sign of the Fichera function at the degenerate boundaries, i.e., if the sign is strictly negative, then boundary conditions are indispensable; otherwise, the reverse is exactly the case. By simple calculations one can easily obtain that the values of the Fichera function of Eq. (1.1) are equivalent to zero on the boundaries $x = 0$ and $x = l$ and thus the boundary conditions should not be given. Therefore, the parabolic problem (1.1) is well-defined.

The mathematical model (1.1) comes from the zero-coupon bond pricing problem arising in finance (see [1,4,5]). For a given coefficient $\rho(t)$, the degenerate parabolic problem (1.1) which is referred as a forward problem consists of the determination of the solution from the given initial condition. By the standard theory for degenerate parabolic equations (see [3]), we known that the degenerate parabolic problem (1.1) is well-posed provided that its coefficients fulfill some appropriate smoothness conditions. However, the inverse Problem P is ill-posed or improperly posed in the sense of Hadamard (see [6,7]). Furthermore, the degree of ill-posedness of Problem P, in a sense, is higher than some other inverse coefficient problems, e.g., the inverse problem of determining the lower coefficient in parabolic equations (see, for instance, [8–11]). The ill-posedness, particularly the numerical instability, is the main difficulty for the inverse problem. In general, since the input data is often obtained by experiments, it is hard to avoid data errors. A sequence of the numerical instability is that arbitrarily small changes in the data may lead to arbitrarily large changes in the solution, which may make the obtained results meaningless (see [12–14]).

Inverse coefficient problems for parabolic equations are well studied in the literature. In [10,9,15–17], an inverse problem of identifying the control parameter $p(t)$ in the following heat conduction equation

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + p(t)w + \phi(x, t), \quad (x, t) \in Q$$

has been considered carefully. A similar problem subject to non-local boundary conditions is studied in [18] by the predictor–corrector method. In [5], the inverse problem of recovering the market price of risk $\lambda(t)$ in the following zero-coupon bond (**ZCB**) equation:

$$\frac{\partial P}{\partial t} + \frac{\omega^2(r)}{2}\frac{\partial^2 P}{\partial r^2} + (\theta(r) + \lambda(t)\omega(r))\frac{\partial P}{\partial r} - rP = 0, \quad (r, t) \in Q \equiv [0, R] \times [0, T)$$

from the current market prices of **ZCB** is discussed. The problem is transformed into a non-linear integral equation and its numerical solution is also obtained.

In [19,20], the authors considered the determination of the implied volatility $\sigma = \sigma(S)$ in the Black–Scholes equation (1.5) from current market prices of options under an optimization control framework. The existence of $\sigma(S)$ and a well-posed algorithm are obtained. Similar results are derived in [21], where a new extra condition, i.e., the average option prices, is assumed to be known. In [22], the existence and uniqueness of determining $a(x)$ in the following parabolic equation

$$v_t - \Delta v + a(x)v = 0, \quad (x, t) \in Q$$

are obtained by the contraction mapping principle. Other treatments regarding these kinds of problems can be found in [8,23,24,11].