Journal of Computational and Applied Mathematics 235 (2011) 2979–2992

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Approximate inverse method for stable analytic continuation in a strip domain☆

Yuan-Xiang Zhang, Chu-Li Fu*, Liang Yan

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China

ARTICLE INFO

Article history:
Received 8 December 2009
Received in revised form 21 October 2010

Keywords:
Analytic continuation
Ill-posed problems
Regularization
Approximate inverse
Reconstruction kernel

ABSTRACT

Numerical analytic continuation is, in general, an ill-posed problem and some special regularization methods are needed. In this paper we apply the approximate inverse method to deal with the problem in a strip domain $\Omega = \{z = x + iy \in \mathbb{C} \mid \ x \in \mathbb{R}, 0 < \mid y \mid < y_0\}$ in complex plane, the data is known approximately only on the line $y = 0$. Error estimate between the exact and approximate solution with regularization parameter selected by both a priori and a posteriori strategy are provided, respectively. Numerical results show that the method works effectively, and can be a competitive alternative to existing method for the problem.

1. Introduction

The problem for analytic continuation of an analytic function is encountered in many practical applications (see, e.g., [1–4] and the references therein), and the numerical analytic continuation is a more interesting and hard problem. It is well known, in general, to be ill-posed in the sense that the solution does not depend continuously on the data. To obtain stable numerical algorithms for ill-posed problems, some effective regularization methods must be adopted, and several nonclassical methods have been developed rapidly in recent years, for example, the Tikhonov-type and Fourier method [5], the mollification method [6], the wavelet method [7], the bounded variations method [8], etc. For different concrete problems, some special methods which can achieve better effect may be used. In the present paper we will use another method—the approximate inverse method to deal with analytic continuation in a strip domain.

In this paper we consider the following problem of analytic continuation again for an analytic function $f(z) = f(x + iy)$ in a strip domain in the complex plane

$$\Omega = \{z = x + iy \in \mathbb{C} \mid x \in \mathbb{R}, |y| \langle y_0, y_0 \rangle 0 \text{ is a positive constant}\}, \tag{1.1}$$

where i is the imaginary unit. The data is only given on the real axis, i.e., $f(z)|_{y=0} = f(x)$ is known approximately and we would extend f analytically from this data to the whole domain Ω. This problem possesses an important practical background, e.g., in medical imaging [9] and therefore it is worth studying. It has been considered in [10] by a mollification regularization method, however, there is no discussion of the numerical implementation of the method in the cited paper. In recently published papers [11,12], the Fourier and a Tikhonov method to solve this problem are proposed, respectively.

In the present paper, we develop a novel approach based on the approximate inverse method to solve the above mentioned problem. The original idea of approximate inverse method can be retrospected to [13], and later in 1990–1999 it was improved in [14,15]. This method can be viewed as a generalization of some linear regularization methods like

☆ The project is supported by the National Natural Science Foundation of China (Nos. 10671085, 10971089).

* Corresponding author.
E-mail address: fuchuli@lzu.edu.cn (C.-L. Fu).

0377-0427/$ – see front matter
doi:10.1016/j.cam.2010.12.017

the Tikhonov–Phillips methods, iterative methods, truncated singular value decomposition, Backus–Gilbert-type methods. All these linear methods can be presented in a unified framework that they can be interpreted as a combination of the pseudo-inverse and smoothing [15]. The approximate inverse method is a very fast method for the reconstruction of solutions for linear and nonlinear ill-posed problems. It has successfully been applied to several areas, such as computerized tomography [16], ultrasound tomography [17] and the inverse scattering problem [18]. In addition, Jonas and Louis [19] applied the approximate inverse in conjunction with Adomian decomposition to a one-dimensional inverse heat conduction problem. For more details about its essential properties and applications, we can refer the reader to [20], and the recently published monograph of Schuster [21].

To our knowledge, so far there is no result about analytic continuation using approximate inverse method. And in comparison with other regularization methods, the theory analysis and numerical implementation of the proposed problem by approximate inverse method is pretty simple. As stated in Remark 2.1, once the reconstruction kernel is known, the regularization approximation can be computed just by a simple evaluation of an inner product.

Before we put an end to this section, we give some preparative knowledge here. Let $\hat{g}$ denote the Fourier transform of function $g(x)$ defined by

$$\mathcal{F}[g](\xi) = \hat{g}(\xi) =: \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ix\xi} g(x) dx, \tag{1.2}$$

and we assume that

$$f(\cdot + iy) \in L^2(\mathbb{R}) \quad \text{for } |y| < y_0. \tag{1.3}$$

Using the Fourier transformation technique with respect to the variable x, it is easy to know [10–12]

$$\mathcal{F}[f(\cdot + iy)](\xi, y) = e^{-y\xi} \hat{f}(\xi), \tag{1.4}$$

or equivalently,

$$f(x + iy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ix\xi} e^{-y\xi} \hat{f}(\xi) d\xi. \tag{1.5}$$

For more details about the solution and the ill-posedness for problem (1.1), we refer to the Refs. [10–12].

Our paper is organized as follows. In Section 2, we compendiously present the method of approximate inverse. In Section 3, a convergence theorem about the method is provided, and error estimates between the exact solution and an approximate solution for both *a* priori and *a* posteriori parameter choice rules are given, respectively. Two numerical examples are given in Section 4. And in the last section we summarize this paper by a conclusion.

2. Approximate inverse

We briefly present the main idea of the approximate inverse, for more details, we can refer to [15,20,21].

Let K be a linear compact operator between the Hilbert spaces X and Y. It is well known that the problem of solving $Kf = g$ is ill-posed in the case of an infinite-dimensional range of K (see, e.g., [22,23]). Therefore, the computation of f with some direct method is unstable. We calculate a smoothed version of the exact solution f of $Kf = g$ or its generalized inverse $f^\dagger = K^\dagger g$ instead of f itself. This is carried out by computing the moments $f_\gamma = \langle f, m_\gamma \rangle_X$, where $\langle \cdot, \cdot \rangle_X$ denote the inner product in X, and m_γ should be chosen appropriately. The subscript γ is the so-called regularization parameter, the proper selection of it is essential for the method.

To overcome the trouble of the computation of f_γ with the unknown exact solution f, we assume for the moment that m_γ is in the range of K^*. Then we have, with ψ_γ being the solution of the equation $K^*\psi_\gamma = m_\gamma$, the relation

$$\langle f, m_\gamma \rangle_X = \langle f, K^*\psi_\gamma \rangle_X = \langle Kf, \psi_\gamma \rangle_Y = \langle g, \psi_\gamma \rangle_Y. \tag{2.1}$$

In the case that m_γ is not in the range of operator K^*, we can calculate ψ_γ as the minimizing function of $\|K^*\psi_\gamma - m_\gamma\|$. This leads to the equation

$$KK^*\psi_\gamma = Km_\gamma. \tag{2.2}$$

Definition 2.1. Let m_γ be a suitable function, and ψ_γ is the solution of (2.2). Then operator S_γ defined by

$$S_\gamma g = \langle g, \psi_\gamma \rangle_Y$$

is called the approximate inverse of the operator K and ψ_γ is the reconstruction kernel associated with m_γ.

Remark 2.1. By (2.2), we can see that the computation of the reconstruction kernel needs only a predetermined exact function m_γ, hence ψ_γ can be precomputed. Moreover, due to (2.1), once the reconstruction kernel is known, the regularization approximation f_γ can be implemented by a simple evaluation of an inner product in Y. More advantages of the method are detailedly described in [14].