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Lyapunov-type inequality for a class of odd-order differential equations

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ABSTRACT

In this paper, we give a generalization of the well-known Lyapunov-type inequality for a class of odd-order differential equations, the result of this paper is new and generalizes some early results on this topic.

1. Introduction

In this paper, we give a generalization of the well-known Lyapunov inequality for the second-order linear differential equation

$$x''(t) + q(t)x(t) = 0 \tag{1}$$

to a class of odd-order linear differential equations. It is well-known [1] that if $q \in C[a, b]$, $x(t)$ is a solution of (1) such that $x(a) = x(b) = 0$, $x(t) \neq 0$ and $t \in (a, b)$, then the following inequality holds:

$$(b - a)\int_a^b |q(t)|\mathrm{d}t > 4 \tag{2}$$

and the constant 4 is sharp, which means that it cannot be replaced by a larger number.

Since this result has found applications in the study of various properties of solutions of differential equation (1) such as oscillation theory, disconjugacy and eigenvalues problems, there are many proofs and generalizations. Such as to nonlinear second order equations, to delay differential equations, to higher order differential equations, to discrete differential equations and to linear Hamiltonian systems. See, for example, the references [1–13] and the references therein. But so far, only a few results have been achieved for odd-order differential equations in this direction. In [11], the authors consider the following third-order linear differential equation:

$$x''' + q(t)x = 0, \tag{3}$$

where $q \in C[a, b]$, they obtain the following result:

Theorem A. *If there exists a $d \in (a, b)$ such that $x''(d) = 0$, then*

$$(b - a)^2\int_a^b |q(t)|\mathrm{d}t > 4. \tag{4}$$

In this paper, we will give a generalization of the above result to a class of odd-order linear differential equations.

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2. Main results

Theorem 1. *Consider the following third-order linear differential equation:*

$$\left(r_2(t)\left(r_1(t)x'\right)'\right)' + q(t)x = 0, \tag{5}$$

where $r_k \in C^{2-k+1}([a,b],(0,+\infty)),\ k = 1, 2, q \in C([a,b],\mathbb{R})$. *If* $x(t)$ *is a nonzero solution of* (5) *satisfying* $x(a) = x(b) = 0$ *and there exists a* $d \in [a,b]$ *such that* $f_1(d) = 0$, *where* $f_1(t) = \left(r_1(t)x'(t)\right)'$. *Then*

$$2\int_a^b |q(t)|\mathrm{d}t > \min_{c\in[a,b]} g_1(c), \tag{6}$$

where

$$g_1(c) = \frac{1}{\int_a^c \frac{\mathrm{d}t}{r_1(t)} \int_a^c \frac{\mathrm{d}t}{r_2(t)}} + \frac{1}{\int_c^b \frac{\mathrm{d}t}{r_1(t)} \int_c^b \frac{\mathrm{d}t}{r_2(t)}}.$$

Theorem 2. *Consider the following linear differential equation of* $2n+1$*-order with* $n > 1$:

$$\left(r_{2n}(t)\left(r_{2n-1}(t)\left(\cdots\left(r_2(t)\left(r_1(t)x'\right)'\right)'\cdots\right)'\right)'\right)' + q(t)x = 0, \tag{7}$$

where $r_k \in C^{2n-k+1}([a,b],(0,+\infty)),\ k = 1, 2, \ldots, 2n, q \in C([a,b],\mathbb{R})$. *If* $x(t)$ *is a nonzero solution of* (7) *satisfying*

$$x^{(k)}(a) = x^{(k)}(b) = 0, \quad k = 0, 1, 2, \ldots, n-1, \tag{8}$$

and the function defined by

$$f_n(t) =: \left(r_{2n-1}(t)\left(\cdots\left(r_2(t)\left(r_1(t)x'\right)'\right)'\right)'\cdots\right)'$$

has a zero $d \in [a,b]$, *then we have*

$$2\left[\prod_{k=n+2}^{2n}\int_a^b \frac{\mathrm{d}t}{r_k(t)}\right]\cdot\int_a^b |q(t)|\mathrm{d}t > \min_{c\in[a,b]} g_n(c),$$

where

$$g_n(c) = \frac{1}{\prod_{j=1}^{n+1}\int_a^c \frac{\mathrm{d}t}{r_j(t)}} + \frac{1}{\prod_{j=1}^{n+1}\int_c^b \frac{\mathrm{d}t}{r_j(t)}}.$$

Corollary 1. . *Assume* $r_1(t) = r_2(t) = r(t) \in C^2([a,b],(0,+\infty)), q \in C([a,b],\mathbb{R})$. *Then* (5) *reduces to*

$$\left(r(t)\left(r(t)x'\right)'\right)' + q(t)x = 0. \tag{9}$$

If a nonzero solution $x(t)$ *of* (9) *satisfies* $x(a) = x(b) = 0$ *and there exists a* $d \in [a,b]$ *such that* $f_1(d) = 0$, *where* $f_1(t) := \left(r(t)x'(t)\right)'$, *then*

$$\left(\int_a^b \frac{\mathrm{d}t}{r(t)}\right)^2\cdot\int_a^b |q(t)|\mathrm{d}t > 4.$$

Corollary 2. *Assume* $n > 1, r_1(t) = \cdots = r_{n+1}(t) = r(t) \in C^{2n}([a,b],(0,+\infty)), q \in C([a,b],\mathbb{R})$. *If a nonzero solution* $x(t)$ *of* (7) *satisfies* $x^{(k)}(a) = x^{(k)}(b) = 0, k = 0, 1, 2, \ldots, n-1$ *and there exists a* $d \in [a,b]$ *such that* $f_n(d) = 0$, *where*

$$f_n(t) =: \left(r_{2n-1}(t)\left(\cdots\left(r_{n+2}(t)\left(r(t)\left(\cdots\left(r(t)x'(t)\right)'\cdots\right)'\right)'\right)'\right)'\cdots\right)',$$

then

$$\left[\prod_{k=n+2}^{2n}\int_a^b \frac{\mathrm{d}t}{r_k(t)}\right]\cdot\int_a^b |q(t)|\mathrm{d}t > \frac{2^{n+1}}{\left(\int_a^b \frac{\mathrm{d}t}{r(t)}\right)^{n+1}}.$$