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New modifications of Potra–Pták's method with optimal fourth and eighth orders of convergence[☆]

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ABSTRACT

In this paper, we present two new iterative methods for solving nonlinear equations by using suitable Taylor and divided difference approximations. Both methods are obtained by modifying Potra–Pták's method trying to get optimal order. We prove that the new methods reach orders of convergence four and eight with three and four functional evaluations, respectively. So, Kung and Traub's conjecture Kung and Traub (1974) [2], that establishes for an iterative method based on n evaluations an optimal order $p = 2^{n-1}$ is fulfilled, getting the highest efficiency indices for orders $p = 4$ and $p = 8$, which are 1.587 and 1.682.

We also perform different numerical tests that confirm the theoretical results and allow us to compare these methods with Potra-Pták's method from which they have been derived, and with other recently published eighth-order methods.

1. Introduction

In recent years, many modified iterative methods for solving nonlinear equations have been developed to improve the local order of convergence of some classical methods such as Newton, Potra–Pták, Chebyshev, Halley and Ostrowski's methods.

As the order of an iterative method increases, so does the number of functional evaluations per step. The efficiency index (see [1]), gives a measure of the balance between those quantities, according to the formula $p^{1/n}$, where p is the order of the method and n the number of function evaluations per step. Kung and Traub [2] conjecture that the order of convergence of any multipoint method without memory cannot exceed the bound 2^{n-1}, (called the optimal order). Thus, the optimal order for a method with 3 function evaluations per step would be 4. Ostrowski's method [1], Jarratt's method [3] and King's method [4] are optimal fourth-order methods, because they only perform three function evaluations per step. Nowadays, obtaining new optimal methods of order 4 is still important, because the corresponding efficiency index, 1.5874, is very competitive. Some papers with this aim have recently appeared in the literature (see [5,6]).

Li et al. presented in [7] a sixteenth order of convergence method, but it uses 6 function evaluations, and so, is not optimal. As far as we know, the most efficient methods studied in the literature are the optimal eighth-order methods with four function evaluations, such as that proposed in [2].

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More recently, some optimal eighth-order methods have been proposed. For example, Bi et al. [8] presented a new family of eighth-order methods based on King's family, that involves Ostrowski's method for a particular value of the parameter $\beta = 0$. This method, to which we will refer as BRW_8, is given by

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - H(\mu_n) \frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n)}, \end{aligned} \tag{1}$$

where $\mu = \frac{f(z_n)}{f(x_n)}$ and $H(t)$ represents a real-valued function. The same strategy, by using the method of weight functions, is used in [9] for introducing a new family of modified Ostrowski's methods with eighth order of convergence. A similar construction of eighth-order methods has been used in [10] starting from a different scheme, by using now $\mu = \frac{f(y_n)}{f(x_n)}$ and introducing a parametric function $H(t)$ in the second step.

Another family of three-point methods of optimal order is introduced by Thukral et al. [11]. The two first steps are the same as in (1), and the last one is

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)} \left(\varphi\left(\frac{f(y_n)}{f(x_n)} \right) + \frac{f(z_n)}{f(y_n) - \beta f(z_n)} + \frac{4f(z_n)}{f(x_n)} \right), \tag{2}$$

where β is a real number and φ is an arbitrary real function satisfying certain conditions. We will refer to this method as TP_8.

One of the last methods with optimal eighth order of convergence can be found in [12]. All of these methods involve the optimal Ostrowski's method and use arbitrary real parameters and weight functions not always easy to determine.

In this paper, we derive optimal order methods of fourth and eighth order starting from the well-known third-order Potra–Pták's method [13], by composing it with modified Newton's iterations and approximating several function evaluations in order to improve the efficiency. The technique used for obtaining the approximations only includes the Taylor expansions and can be expressed in terms of divided differences.

The rest of this paper is organized as follows: Section 2 describes our methods, in Section 3 we establish the convergence order of the new methods. Finally, in Section 4, different numerical tests confirm the theoretical results and allow us to compare these variants with other known methods.

2. Description of the methods

Let us consider the nonlinear equation $f(x) = 0$, where $f(x)$ is a function of class $n + 1$ defined on a real interval I. If $x_0, x_1, \ldots, x_n$ are points of I the divided difference of order 0 is:

$$f[x_0] = f(x_0),$$

and the divided differences of order 1 can be expressed by:

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}.$$

In general, the divided difference of order n is obtained as:

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, x_n] - f[x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}.$$

If $f \in \mathcal{C}^{n+1}(I)$ then we have:

$$f[x_0, x_1, \ldots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!},$$

for a suitable $\xi \in I$.
In the limit, for $x_0 = x_1 = \cdots = x_{n+1} = x$, one can write:

$$f[x, x, \ldots, x] = \frac{f^{(n+1)}(x)}{(n+1)!}. \tag{3}$$

We use these properties in order to approximate the high-order derivatives which appear in the new iterative method of optimal order eight.