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Functionals of exponential Brownian motion and divided differences

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ABSTRACT

We provide a surprising new application of classical approximation theory to a fundamental asset-pricing model of mathematical finance. Specifically, we calculate an analytic value for the correlation coefficient between the exponential Brownian motion and its time average, and we find that the use of divided differences greatly elucidates formulae, providing a path to several new results. As applications, we find that this correlation coefficient is always at least $1/\sqrt{2}$ and, via the Hermite–Genocchi integral relation, demonstrate that all moments of the time average are certain divided differences of the exponential function. We also prove that these moments agree with the somewhat more complex formulae obtained by Oshanin and Yor.

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1. Introduction

We begin with exponential, or geometric, Brownian motion, defined by

$$S(t) = e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma B(t)}, \quad t \geq 0, \tag{1.1}$$

where r and σ are non-negative constants, and $B : [0, \infty) \to \mathbb{R}$ is Brownian motion. In other words, B is a stochastic process, or *random function*, for which $B(0) = 1$, its increments are independent, and for $0 \leq s < t$, the increment $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$. The basic properties of Brownian motion are explained in Section 37 of Billingsley [1], while Karatzas and Shreve [2] is a comprehensive treatise.

We shall study the *time average*:

$$A(T) := \frac{1}{T}\int_0^T S(t)\,dt \tag{1.2}$$

using the calculus of divided differences, a fundamental tool in approximation theory. We will, in particular, show that the correlation coefficient between $A(T)$ and $S(T)$, the moments of $A(T)$ and, more generally, joint moments of $S(T)$ and $A(T)$ can be elegantly, and usefully, expressed in terms of divided differences of the exponential function. Now the time average $A(T)$ has been extensively studied in the literature of Asian options; see, for instance, [3,4]. However, we find that our use of divided differences both simplify and elucidate formulae. In Section 2, we derive the correlation coefficient for $S(T)$ and $A(T)$, finding that it is always at least $1/\sqrt{2}$, thus explaining the relative high correlation that is observational folklore in the financial community. In Section 3, we demonstrate that the divided differences occurring in the lower moments of $S(T)$ and $A(T)$ generalise to all moments, using the fact that the integral of an exponential function over a simplex can be expressed, via the Hermite–Genocchi formula, as a certain divided difference of the exponential function. In Section 4, we provide the

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divided difference theory required by the paper. Finally, in Section 5, we use our divided difference approach to derive a recurrence relation for the moments of $A(T)$.

We first observe the familiar result

$$\mathbb{E}S(T) = e^{(r-\sigma^2/2)T}\mathbb{E}e^{\sigma T^{1/2}Z} = e^{(r-\sigma^2/2)T}e^{\sigma^2 T/2} = e^{rT}. \tag{1.3}$$

Here Z denotes a generic $N(0, 1)$ Gaussian random variable and we have used the standard fact that

$$\mathbb{E}e^{\lambda Z} = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{\lambda\tau}e^{-\tau^2/2}\,d\tau = (2\pi)^{-1/2}\int_{\mathbb{R}} e^{-\frac{1}{2}\{(\tau-\lambda)^2-\lambda^2\}}\,d\tau = e^{\lambda^2/2}. \tag{1.4}$$

Similarly,

$$\begin{aligned}\mathbb{E}A(T) &= T^{-1}\int_0^T \mathbb{E}S(t)\,dt \\ &= \frac{e^{rT}-1}{rT}.\end{aligned} \tag{1.5}$$

The approximation theorist will immediately recognise the divided difference

$$\mathbb{E}A(T) = \exp[0, rT], \tag{1.6}$$

but a sceptical reader might view this as mere coincidence; in fact, it is but the tip of an iceberg. We remind the reader that $f[a_0, a_1, \ldots, a_n]$ is the highest coefficient of the unique polynomial of degree n interpolating f at distinct points $a_0, \ldots, a_n \in \mathbb{R}$, which implies $f[a_0] = f(a_0)$ and

$$f[a_0, a_1] = \frac{f(a_1) - f(a_0)}{a_1 - a_0}.$$

Further, it is evident that a divided difference does not depend on the order in which the points $a_0, a_1, \ldots, a_n$ are chosen. As mentioned above, Section 4 collects further divided difference theory required by this paper.

2. The correlation coefficient between the time average and the asset

We shall compute the correlation coefficient between $S(T)$ and $A(T)$. Specifically, we calculate

$$R := \frac{\mathbb{E}\left(S(T)A(T)\right) - \mathbb{E}\left(S(T)\right)\mathbb{E}\left(A(T)\right)}{\sqrt{\operatorname{var} S(T)\operatorname{var} A(T)}}. \tag{2.1}$$

We find an elegant divided difference expression for R.

Theorem 2.1. *The correlation coefficient (2.1) is given by*

$$R \equiv R(rT, \sigma^2 T) = \frac{\exp[rT, 2rT, (2r+\sigma^2)T]}{\sqrt{2\exp[2rT, (2r+\sigma^2)T]\exp[0, rT, 2rT, (2r+\sigma^2)T]}}. \tag{2.2}$$

Let us begin our derivation.

Lemma 2.2. *If* $0 \le a \le b$, *then*

$$\mathbb{E}S(a)S(b) = \exp\Big(a(r+\sigma^2) + br\Big). \tag{2.3}$$

Proof. We have

$$\begin{aligned}\mathbb{E}S(a)S(b) &= \mathbb{E}S(a)^2 e^{(b-a)(r-\sigma^2/2)+\sigma(B(b)-B(a))} \\ &= \mathbb{E}S(a)^2\mathbb{E}e^{(b-a)(r-\sigma^2/2)+\sigma\sqrt{b-a}Z} \\ &= e^{(2r+\sigma^2)a}e^{(b-a)r} \\ &= e^{a(r+\sigma^2)}e^{br},\end{aligned} \tag{2.4}$$

where $Z \sim N(0, 1)$ and we have used (1.3). □

Proposition 2.3. *We have*

$$\mathbb{E}S(T)A(T) = \exp[rT, (2r+\sigma^2)T]. \tag{2.5}$$

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